# 4 Using Gaussian elimination: Column space, nullspace, rank, nullity, linear independence, inverse matrix

## 4.1 An example

The following example will be used to illustrate the concepts. Let

$$A = \begin{pmatrix} 1 & 2 & -1 & 3 & 0 \\ -1 & -2 & 2 & -2 & -1 \\ 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 2 & 2 & -1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 6 \\ 7 \end{pmatrix}$$

(recall the convention: this is an $n \times k$ matrix with $n = 4$ and $k = 5$).

**Problem 4.1** *Find the full solution to* $A\mathbf{x} = \mathbf{b}$

SOLUTION: The row echelon form of $[A|\mathbf{b}]$ is (CHECK (*))

$$\text{Row echelon} = \left(\begin{array}{ccccc|c} \underline{1} & 2 & -1 & 3 & 0 & 1 \\ 0 & 0 & \underline{1} & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & \underline{1} & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{4.1}$$

(pivots underlined). The reduced row echelon form

$$\text{Reduced row echelon} = \left(\begin{array}{ccccc|c} \underline{1} & 2 & 0 & 4 & 0 & 6 \\ 0 & 0 & \underline{1} & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & \underline{1} & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{4.2}$$

Hence the full solution is

$$\mathbf{x} = \begin{pmatrix} 6 \\ 0 \\ 5 \\ 0 \\ 3 \end{pmatrix} + s \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -4 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \tag{4.3}$$

with free parameters $s = x_2$ and $t = x_4$.

## 4.2 Definitions

First recall the definitions. Let $A$ be an $n \times k$ matrix.

**Definition 4.2** *The linear subspace of* $\mathbf{R}^n$ *spanned by the columns of* $A$ *is called the* **column space** *of* $A$. *It is also called the range or the image of* $A$. *It is denoted by* $R(A)$ *or* $Im(A)$ *(we will use* $R(A)$*) and it can be written as*

$$R(A) := \{A\mathbf{x} \ : \ \mathbf{x} \in \mathbf{R}^k\}$$

*i.e. as the set of all linear combinations of the columns. The dimension of* $R(A)$ *is the* **rank** *of* $A$.

**Definition 4.3** *The set of elements* $\mathbf{x} \in \mathbf{R}^k$ *which solve* $A\mathbf{x} = 0$ *is called the* **null-space** *of* $A$ *and is denoted by* $N(A)$*:*

$$N(A) := \{\mathbf{x} \in \mathbf{R}^k \ : \ A\mathbf{x} = 0\}.$$

*It is easy to see (CHECK(*)) that* $N(A)$ *is a linear subspace of* $\mathbf{R}^k$. *Its dimension is called the* **nullity** *of* $A$.

Of course one has to check that these are really subspaces (i.e. closed under addition and scalar multiplication). This is left as an exercise (*).

The relation of these sets to the solutions of

$$A\mathbf{x} = \mathbf{b} \tag{4.4}$$

is the following:

- The column space is used to decide whether (4.4) is solvable or not. This equation is solvable if and only if $\mathbf{b} \in R(A)$.

- The null space is used to decide how many solutions you have, once you know that you have at least one, i.e. $\mathbf{b} \in R(A)$. If the nullspace is trivial, $N(A) = \{\mathbf{0}\}$, then there is exactly one solution. If the nullspace is nontrivial, then you have free parameters in the general solution to (4.4). The number of free parameters equals to the nullity of $A$.

- The set of solutions to $A\mathbf{x} = \mathbf{b}$ is an affine set ("affine subspace"). It has the form $\mathbf{x} = \mathbf{x}_0 + N(A) = \{\mathbf{x}_0 + \mathbf{y} \; : \; \mathbf{y} \in N(A)\}$ where $\mathbf{x}_0$ is any fixed solution. In other words, the solution set is an affine set which is a translate of $N(A)$. The translating vector $\mathbf{x}_0$ is certainly not unique.

How to recognize these sets from the Gauss elimination? You can ask this question in two different ways. You may want to find a basis in both spaces, i.e. represent these spaces as a linear combination of a (minimal) spanning set. But you may be interested in the so called "membership" problem, i.e. given a vector, you want to decide whether it belongs to the space or not. This latter is especially important for the column space.

## 4.3 Basic problems

**Problem 4.4** *Does a given vector* $\mathbf{b}$ *belong to* $R(A)$ *or not?*

SOLUTION: Run the Gauss elimination for the extended matrix $[A|\mathbf{b}]$. If the last column has a pivot element, then $A\mathbf{x} = \mathbf{b}$ is not solvable, i.e. $\mathbf{b} \notin R(A)$. Otherwise $\mathbf{b} \in R(A)$.

EXAMPLE: In the example in Section 4.1 the last column (the column of $\mathbf{b}$) in the echelon matrix does not contain pivot. Hence the equation is solvable. For this purpose it does not matter whether you look at the row echelon or reduced row echelon form.

If the original $\mathbf{b}$ were changed to $\mathbf{b}' = \begin{pmatrix} 1 \\ 1 \\ 6 \\ 2 \end{pmatrix}$, then the row echelon form (4.1) would be changed to (CHECK (*))

$$\left(\begin{array}{ccccc|c} \underline{1} & 2 & -1 & 3 & 0 & 1 \\ 0 & 0 & \underline{1} & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & \underline{1} & 3 \\ 0 & 0 & 0 & 0 & 0 & \underline{1} \end{array}\right).$$

Since there is a pivot in the last column, the equation $A\mathbf{x} = \mathbf{b}'$ has no solution.

**Problem 4.5** *Find the constraint equations for the column space $R(A)$ of a given matrix. In other words, find equations for the coordinates of $\mathbf{b} \in R(A)$ (general membership problem).*

SOLUTION: This is essentially the previous problem. Run the Gauss elimination with a general vector $\mathbf{b}$. The constraint equations are found in the last entry of those rows of the eliminated augmented matrix $[A|\mathbf{b}]$ whose all other entries are zero.

EXAMPLE: In the example in Section 4.1 we have to run the elimination for the matrix

$$A = \left(\begin{array}{ccccc|c} 1 & 2 & -1 & 3 & 0 & b_1 \\ -1 & -2 & 2 & -2 & -1 & b_2 \\ 1 & 2 & 0 & 4 & 0 & b_3 \\ 0 & 0 & 2 & 2 & -1 & b_4 \end{array}\right)$$

with a general last column. The row echelon form is

$$\text{Row echelon} = \left(\begin{array}{ccccc|c} \underline{1} & 2 & -1 & 3 & 0 & b_1 \\ 0 & 0 & \underline{1} & 1 & -1 & b_1 + b_2 \\ 0 & 0 & 0 & 0 & \underline{1} & -2b_1 - b_2 + b_3 \\ 0 & 0 & 0 & 0 & 0 & -b_2 - b_3 + b_4 \end{array}\right)$$

Hence the constraint for $\mathbf{b} \in R(A)$ is

$$-b_2 - b_3 + b_4 = 0.$$

If you have more fully zero rows on the left of the vertical line, then the constraint is a system of homogeneous equations for $\mathbf{b}$, and the coordinates of $\mathbf{b}$ must satisfy all of them in order to be in $R(A)$.

**Problem 4.6** *Find a basis in $R(A)$.*

SOLUTION: The pivotal columns of $A$ form a basis for $R(A)$, i.e. those columns in $A$ which correspond to the columns in the row-echelon form of $A$ (and not in $[A|\mathbf{b}]$!) containing the leading ones. The dimension of $R(A)$ hence is the number of pivot columns.

*Proof:* See theorem 2.5.5 in [HH]. The idea is that the pivotal columns in the row echelon (or reduced row echelon) form clearly span the column space of the echelon matrix (WHY? (*)) and they are independent (WHY? (*)). Moreover the spanning property remains invariant under the elementary operations. $\square$

Be CAREFUL: you have to pick the columns from the original matrix and not from the echelon matrix. The echelon matrix just tells you *which* columns to pick. For this purpose you can use both the row-echelon matrix or the reduced row-echelon matrix (the leading ones are the same in both), and you can forget about the the column of $\mathbf{b}$.

EXAMPLE: In the example in Section 4.1 a basis of $R(A)$ is

$$\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} -1 \\ 2 \\ 0 \\ 2 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \end{pmatrix}$$

i.e. the first, third and fifth columns of $A$, since these columns contained pivots in (4.1).

**Problem 4.7** *Find a basis in $N(A)$.*

SOLUTION: Gauss elimination gives the full solution to any equation $A\mathbf{x} = \mathbf{b}$, in particular you can choose $\mathbf{b} = \mathbf{0}$. It means that the augmented matrix has a zero last column $[A|\mathbf{0}]$, and we can safely forget about it, just do the elimination for $A$ up to the reduced row echelon form. Let $k_1, k_2, \dots k_p$ be the positions of the nonpivotal columns (index of the free variables). Choose the vectors $\mathbf{v}_i \in \mathbf{R}^k$ for $i = 1, 2, \dots p$ such that its $k_i$-th entry be one, all other $k_j$-th entries be zero and solve $A\mathbf{v}_i = 0$. Such vectors form a basis of $N(A)$. The dimension of $N(A)$ is clearly the number of nonpivotal columns, or the number of free variables.

For the proof see Theorem 2.5.7 in [HH]. The idea is to write the full solution to $A\mathbf{x} = \mathbf{0}$ as a linear combination of $p$ vectors with the $p$ free parameters as coefficients. Recall that when solving a linear system of equations, the result of the Gaussian elimination is exactly in this form. Then one can show that these $p$ vectors are linearly independent, since each contains a 1 at a place where all others have zero entry.

EXAMPLE: In the example in Section 4.1 a basis of $N(A)$ is

$$\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} -4 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \tag{4.5}$$

since these are the vectors whose coefficients are the free parameters. Notice that each vector contains a 1 at the place of the corresponding free parameter and the other vector has zero entry at that slot. This structure follows from the way how you obtained these vectors from (4.2).

**Problem 4.8** *Given vectors $\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_k \in \mathbf{R}^n$. Are they linearly independent or dependent?*

SOLUTION: Form the $n \times k$ matrix $A = [\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_k]$ from these vectors as its columns. The vectors are linearly independent if and only if $N(A) = \{\mathbf{0}\}$, i.e. the nullspace is trivial. This is because the vector $A\mathbf{x}$ is exactly the linear combination of the given vectors with coefficients $x_1, x_2, \ldots x_k$. The vectors are linearly independent if only the trivial linear combination gives the zero vector. In other words, if the equation $A\mathbf{x} = \mathbf{0}$ has only trivial solution, i.e. the nullspace of $A$ is trivial.

In practical terms: after forming the matrix $A$, run the Gauss elimination. If there is a nonpivot column, then the original vectors are linearly dependent (and the nontrivial choice of the free variable gives a nontrivial linear dependence among these vectors). If all columns are pivot columns, then the original vectors are lin. independent.

EXAMPLE: In the example in Section 4.1 the columns are not linearly independent since there are nonpivot columns. The nontrivial elements of the nullspace give a nontrivial linear combination of the columns, e.g. from the first basis vector of $N(A)$ in (4.5) you can read off that

$$(-2)\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 2 \\ -2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

or from the second basis vector in (4.5) you can read off that

$$(-4)\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + (-1)\begin{pmatrix} -1 \\ 2 \\ 0 \\ 2 \end{pmatrix} + 1\begin{pmatrix} 3 \\ -2 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

These are just examples of nontrivial linear combinations among the column vectors.

**Problem 4.9** *Given a vector* $\mathbf{b}$ *and a set of other vectors* $\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_k}$ *in* $\mathbf{R}^n$*. Write* $\mathbf{b}$ *as a linear combination of these vectors (if possible)*

SOLUTION: Again form the $n \times k$ matrix $A = [\mathbf{v}_1 \mathbf{v}_2 \ldots \mathbf{v}_k]$ from these vectors as its columns. Then the solution to $A\mathbf{x} = \mathbf{b}$ (if exists) exactly gives you the coefficients $x_1, x_2, \ldots x_k$ with which you have to form the linear combination of the $\mathbf{v}$'s to get $\mathbf{b}$. This is really just a reformulation of the original problem of solving $A\mathbf{x} = \mathbf{b}$.

## 4.4 Dimension formula (Law of conservation of dimensions)

From Problem 2 and 3 above we see that

$$\text{rank}(A) = \dim R(A) = \#\text{pivot columns}$$

$$\text{nullity}(A) = \dim \text{Ker}(A) = \#\text{non-pivot columns}$$

Hence **Dimension formula**

$$\text{rank}(A) + \text{nullity}(A) = \#\text{columns} = k. \tag{4.6}$$

In the example in Section 4.1 we have that $\text{rank}(A) = 3$, $\text{nullity}(A) = 2$ and clearly $3 + 2 = 5$ is the number of columns.

The dimension formula has several consequences. For example we can easily give another proof of Theorem 3.3 (although this is a bit cheating since in our setup Theorem 3.3 was already used to show that basis exists)

**Corollary 4.10** *Let $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_k$ be vectors in $\mathbf{R}^n$. If $k > n$ (i.e. you have more vectors than their size), then these vectors are linearly dependent.*

*Proof:* Since $R(A)$ is a subspace of $\mathbf{R}^n$, its dimension is at most $n$, i.e. $\text{rank}(A) \leq n$. Hence it follows from (4.6) that $\text{nullity}(A) \geq k - n \geq 1$, i.e. $N(A)$ is nontrivial. Hence the columns are linearly dependent. $\square$

REMARK: This statement implies in particular, that if you have more unknowns in a linear system of equations than equations, then you do not have unique solution. WARNING: It does not mean that you have solution at all! All it says is that IF you have at least one solution, then you have infinitely many.

**Corollary 4.11** *Let $A$ be an $n \times n$ square matrix. Then $A\mathbf{x} = \mathbf{b}$ has solution for all $\mathbf{b}$ if and only if the only solution to $A\mathbf{x} = \mathbf{0}$ is the zero vector (in other words, $A$ has linearly independent columns, or $A$ has full rank).*

*Proof:* From (4.6) it is clear that in case of $k = n$ (square matrix), the condition that $\text{rank}(A) = n$ (full rank) and that $\text{nullity}(A) = 0$ are equivalent. □

## 4.5 Relation between the "column-rank" and "row-rank"

The rank is defined as the maximal number of linearly independent columns. This is really a "column-rank", since it focuses on the columns of the matrix $A$. One could equally ask for the maximal number of linearly independent rows of $A$, which would be the "row-rank" of the matrix (be careful: the "row-rank" is NOT the nullity). Notice that the "row-rank" of $A$ is clearly the same as the "column-rank" of the transpose $A^t$. The following fact is a basic result:

**Theorem 4.12** *For any matrix $A$*

$$\mathit{rank}(A) = \mathit{rank}(A^t) \tag{4.7}$$

*i.e. the "column-rank" and the "row-rank" coincide. This common number is the number of pivot elements. In particular the number of pivot elements in the echelon forms of $A$ and $A^t$ coincide (though these forms look completely different, even their sizes are different!!)*

*First proof of Theorem 4.12*: Prop. 2.5.12 in [HH]. The basic idea is that the row space of $A$ and the row space of the reduced row echelon form $\widetilde{A}$ of $A$ are the same. It is clear that the rows of $\widetilde{A}$ can be represented as linear combinations of the rows of $A$ since this is exactly what the elementary steps did. But notice that the elementary operations are reversible; you can go *back* from $\widetilde{A}$ to $A$ again by elementary operations. Hence the rows of $A$ are expressible as linear combinations of the rows of $\widetilde{A}$.

But it is clear that the dimension of the row space of $\widetilde{A}$ is the same as the number of pivots: the nonzero rows in $\widetilde{A}$ are linearly independent, since each contain a 1 at a place where all other row has entry zero. Finally, the number of pivots is the same as rank$(A)$ (see Problem 4.6), which proves (4.7). $\square$

We give another proof of this theorem which uses the following

**Lemma 4.13** *The spaces $N(A)$ and $R(A^t)$ are orthogonal complements of each other within $\mathbf{R}^k$. This means that any vector from $N(A)$ is orthogonal to any vector from $R(A^t)$, and the vectors in these two spaces span $\mathbf{R}^k$.*

*Proof:* (See [D] Theorem 1. p. 79) Pick a vector $\mathbf{x} \in N(A)$, i.e. $A\mathbf{x} = \mathbf{0}$. Hence for any vector $\mathbf{y} \in \mathbf{R}^n$ we have $\mathbf{y}^t \cdot A\mathbf{x} = \mathbf{0}$. Taking the transpose of this relation, we have $\mathbf{x}^t \cdot A^t\mathbf{y} = \mathbf{0}$. This is true for any $\mathbf{y}$, hence $\mathbf{x}$ is orthogonal to the whole column space of $A^t$.

Suppose now that a vector $\mathbf{x} \in \mathbf{R}^k$ is orthogonal to $R(A^t)$. Then, exactly as before, we conclude that $\mathbf{x} \in N(A)$.

From these two statements not just the orthogonality of $R(A^t)$ and $N(A)$ follows, but their spanning property as well. To show that they actually span $\mathbf{R}^k$, we suppose, on contrary, that they don't and we try to get a contradiction. Consider the span of $N(A)$ and $R(A^t)$, if this is not the whole $\mathbf{R}^k$, then there is a nonzero vector $\mathbf{x} \in \mathbf{R}^k$ which is orthogonal to both $N(A)$ and $R(A^t)$. But we have seen that if a vector is orthogonal to $R(A^t)$, then it must be in $N(A)$, hence it cannot be orthogonal to that. This contradiction shows that the span of $N(A)$ and $R(A^t)$ is the whole $\mathbf{R}^k$. $\square$

From this lemma, Theorem (4.12) easily follows.

*Second proof of Theorem (4.12)* Choose a basis $\mathbf{v}_1, \ldots, \mathbf{v}_p$ in $N(A)$ and a basis $\mathbf{w}_1, \ldots, \mathbf{w}_r$ in $R(A^t)$. Here $p$ is the nullity of $A$ and $r$ is the rank of $A^t$. Consider the union of these two bases, i.e. the set $\left\{\mathbf{v}_1, \ldots, \mathbf{v}_p, \mathbf{w}_1, \ldots, \mathbf{w}_r\right\}$. It is easy to see that this is a linearly independent set (WHY? (*) Hint: mimic the proof of Lemma 2.6), hence it is a basis in the span of $R(A^t)$ and $N(A)$, which is $\mathbf{R}^k$. Therefore $p + r = k$, i.e.

$$\text{nullity}(A) + \text{rank}(A^t) = k$$

From (4.6) we have

$$\text{rank}(A) + \text{nullity}(A) = k$$

and from these two relations (4.7) follows. $\square$

A different presentation of the relation between various ranks can be found on `http://www.math.gatech.edu/ carlen/1502/html/pdf/rank.pdf`

## 4.6 Inverse matrix

**Definition 4.14** *Let $A$ be an $n \times n$ square matrix (only square matrices have inverses). The matrix $B$ satisfying $BA = AB = I$ is called the inverse of $A$ and is denoted by $B = A^{-1}$*

REMARK: It is enough to require that $AB = I$ (right inverse) **OR** that $BA = I$ (left inverse), the other one will follow. This is a quite nontrivial fact (since matrix multiplication is noncommutative, i.e. in general $AB = BA$ is not true), and the convenient notation hides it. For its proof, first notice that *if* you have both right and left inverses (i.e. you have matrices $B, C$ such that $AB = I$ and $CA = I$), then $B = C$ easily follows (WHY?(*) Hint: compute

$CAB$). But how do you know that left inverse ($C$) exists if you know only that the right inverse ($B$) exists? As the problem below shows, $B$ exists if $A$ has full rank. But then, by Theorem (4.12), you know that $A^t$ has full rank, too. Hence $A^t$ has a right inverse, say $D$, i.e. $A^tD = I$. But then $C := D^t$ is a left inverse of $A$, i.e. $CA = I$ (WHY? (*)). $\square$

**Problem 4.15** *Compute $A^{-1}$ for a given $A$.*

SOLUTION: Consider the "double" matrix $[A|I]$, and perform a row reduction. $A$ is invertible if and only if the reduced row echelon form of $[A|I]$ is of the form $[I|B]$ with some matrix $B$ (i.e. each column of $A$ has a pivot). In this case $B$ is the inverse matrix.

*Proof*: Finding $B$ such that $AB = I$ really amounts to finding the columns of $B = [\mathbf{v}_1\ \mathbf{v}_2\ \ldots \mathbf{v}_n]$ such that $A\mathbf{v}_j = \mathbf{e}_j$ for all $j = 1, 2, \ldots n$. When row reducing $[A|I]$ you actually solve these $n$ systems of equations simultaneously.

**Theorem 4.16** *The following facts are equivalent for an $n \times n$ square matrix $A$:*

*(i) $A$ is invertible*

*(ii) There are exactly $n$ pivot elements in the reduced row echelon form of $A$.*

*(iii) $N(A) = \{\mathbf{0}\}$*

*(iv) rank$(A) = n$*

*(v) The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b} \in \mathbf{R}^n$. This solution is given by $\mathbf{x} = A^{-1}\mathbf{b}$.*

*(vi) The determinant of $A$ is non-zero.*

*A matrix satisfying these properties is called* **regular** *or* **nonsingular**.

REMARK 1.: It seems tempting to solve a linear system of equations $A\mathbf{x} = \mathbf{b}$ by simply computing $A^{-1}$ and then taking $\mathbf{x} = A^{-1}\mathbf{b}$. Theoretically this is perfect if the matrix is invertible. However, there are two troubles with it:

(a) One would like to solve nonsquare or degenerate equations as well, when you expect many solutions. In this case $A^{-1}$ does not exist (for rectangular matrices it is even not defined in this way) and not necessarily because there is no solution; rather the opposite. The inverse of a singular (square) matrix does not exists because for some $\mathbf{b}$ there is no solution, for some other $\mathbf{b}$ there are infinitely many. Nonexistence of the inverse matrix does not mean that $A\mathbf{x} = \mathbf{b}$ has no solution!

(b) Computing all entries of $A^{-1}$ is a big work. If you solve $A\mathbf{x} = \mathbf{b}$ directly, it is much cheaper. Of course if you have to solve this equation for many different $\mathbf{b}$, then computing $A^{-1}$ could pay off. But remember that the Gaussian elimination also has a tool (Section 8.2 in the Appendix) to facilitate this problem and avoid redoing the same elimination. This method is usually faster than computing $A^{-1}$.

*Proof of Theorem 4.16.* We prove the equivalence of the first five statements cyclically:

$(i) \Longrightarrow (v)$: If $A^{-1}$ exists, then $\mathbf{x} = A^{-1}\mathbf{b}$ makes sense. It is clear that this vector solves the equation, since $A\mathbf{x} = A(A^{-1}\mathbf{b}) = AA^{-1}\mathbf{b} = I\mathbf{b} = \mathbf{b}$.

$(v) \Longrightarrow (iv)$. If the equation $A\mathbf{x} = \mathbf{b}$ solvable for all $\mathbf{b}$, then $A$ must have full rank.

$(iv) \Longrightarrow (iii)$ See the Dimension formula (4.6)

$(iii) \Longrightarrow (ii)$ Nullspace is trivial if there are no free variables, i.e. every column contains a pivot.

$(ii) \Longrightarrow (i)$ The $n$ pivot elements guarantee that the reduction of the enlarged matrix $[A|I]$ ends up with $[I|B]$. As we have seen, $B$ is the inverse matrix. $\square$

We will not prove rigorously that condition (vi) is also equivalent to all the others, since it relies on Cramer's formula and on the definition and some properties of the determinants listed in the "Steps to compute determinants" at the end of Section 9.1. We did not prove them, but all of them follow from elementary arithmetics.

Here we just remark that accepting Cramer's formula (Section 9.2), it is clear that if $\det(A) \neq 0$, then for any $\mathbf{b}$ there exists a solution to $A\mathbf{x} = \mathbf{b}$, i.e. (iv) follows. For the opposite statement, we have to use that the fact whether a determinant is zero or not does not change under the Gaussian elimination steps. For this statement see the "Steps to compute determinants" at the end of Section 9.1. Accepting this, it is clear that if $\det(A) = 0$, then there could not be pivots in all columns after the elimination, since then the determinant of the reduced matrix would be 1 (since it is upper triangular, so the determinant is the product of the diagonal elements). Hence $\det(A) \neq 0$ if and only if the matrix is regular.