8 Appendix 1. Gaussian Elimination in details

The purpose of this appendix is to clarify two things in the book [D]:

1, Choice of the free variables (section 1.2 of [D])

2, Recording the steps for later use (section 1.3 of [D])

There is nothing wrong in the book and you can follow it, this appendix is meant for extra help.

You should keep in mind that there are several algorithmic procedures to implement Gaussian elimination, all have the same basic idea. For actual computational purposes some of them could be more convenient than the others.

We recall that [D] does not insist on ones in the pivot positions, but this can always be achieved by dividing through the rows by the pivot.

We also recall that the book [D] finishes the elimination at the row-echelon form and not at the reduced row-echelon form. See Section 3 for explanation.

8.1 How to determine the free variables in [D] Section 1.2?

Gaussian elimination consists of two steps:

a, transform into triangular form;

b, solve the triangular system.

The triangular system corresponding to a given system of equations is not at all unique. I suggest the following algorithmic procedure (but many other ways are correct as well).

Step 1. Order the unknowns, i.e. give them indices, like $x_1, x_2, \ldots$ and write all equations following this order. In most problems the equations are already given in this form. But if you are given for example

$$\begin{array}{ccccccc} 2x_1 & + & 3x_2 & - & x_3 & = & 3 \\ x_2 & + & x_1 & + & x_3 & = & 2 \\ x_1 & - & x_2 & & & = & 4 \end{array}$$

then it is better to rewrite it as

$$\begin{array}{ccccccc} 2x_1 & + & 3x_2 & - & x_3 & = & 3 \\ x_1 & + & x_2 & + & x_3 & = & 2 \\ x_1 & - & x_2 & & & = & 4 \end{array}$$

right at the beginning.

Step 2. Choose an equation which contains x_1 with nonzero coefficient (if there is no such equation, then x_1 is completely free, it can be anything). By interchanging equations, put this equation into the first line (this is just for convenience). Use this equation (called *pivot equation for* x_1) to eliminate x_1 from all but the pivot equation. This you can do by subtracting appropriate multiples of the pivot equation from all the other equations (the multiple which you use to eliminate x_1 from a given equation is

$$\frac{\text{coeff. of } x_1 \text{ in the given equation}}{\text{coeff. of } x_1 \text{ in the pivot equation}}.$$

You never change the pivot equation!!

In order to simplify your life, you can choose that equation for the pivot equation which contains the "simplest" coefficient of x_1, e.g. 1 or -1, or a small number (Funnily enough, sometimes this could be the *worst* choice for the purpose of numerical stability as we will discuss later. If you implement this on a computer and you face with round-off errors, then dividing with small numbers is always dangereous. So then you'd better choose the biggest - in absolute value - coefficient. But in this class, for practice and for explicit calculations with reasonable rational numbers, the smallest coefficient is usually more comfortable, leading to fractions with fairly small denominators).

In the above example, choose for example the second equation as a pivot equation for x_1, so interchange the order of equations

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 2 \\ 2x_1 & + & 3x_2 & - & x_3 & = & 3 \\ x_1 & - & x_2 & & & = & 4 \end{array}$$

and now declare the first equation as pivot for x_1. By subtracting twice the first equation from the second and then the first from the last, you eliminate all x_1's (apart from the pivot equation for x_1):

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 2 \\ & & x_2 & - & 3x_3 & = & -1 \\ & - & 2x_2 & - & x_3 & = & 2 \end{array}$$

Step 3.

Now you have a system where only the first equation contains x_1, all the rests "start" from x_2 (or higher). Choose one of these equations which contains x_2 with nonzero coefficient. Again, for convenience, try to choose the one where this coefficient is "simple". This equation will be the pivot equation for x_2 which you use to kill x_2 in all the other equations *below* it, i.e. you do not touch the first equation anymore. By interchanging the order of equations, write the pivot equation for x_2 into the second line, below the first equation, and fix it there (you don't change it any more). Then, by subtracting an approriate multiple of the pivot equation for x_2 from the all the equations below it, you can eliminate x_2 from all these equations (but don't forget to keep the pivot equation at its place in the second row). You end up with a system where the first equation might have all variables, the second does not have x_1, and all the rest does not have x_1 and x_2.

If there is no equation (apart from the first one) which contains x_2, then x_2 will be a free variable.

Continuing with the example above, you choose the second equation as pivot for x_2, and subtract (-2) times of it from the last one (which of course means adding twice the second to the last). You get

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 2 \\ & & x_2 & - & 3x_3 & = & -1 \\ & & & - & 7x_3 & = & 0 \end{array}$$

Notice that the first equation is untouched.

Step 4. Repeat Step 3 with x_3, then with x_4 etc. instead of x_2. When pivoting x_3, you don't touch the first two equations any more etc.

Finally you end up with a triangular form in a sense, that each equation contains at least one less variable than the previous one. The above example shows the typical case, when each equation has one less variable, and the number of equations is the same as the number of variables. In this case you can simply solve it backwards (in this example, first you get $x_3 = 0$ from the last equation, then $x_2 = -1$ from the second and $x_1 = 3$ from the first).

But in general you might end up with a triangular system where there could be extra equations (all with zero coefficients) or you might have less equations than unknowns. Then the procedure is the following:

Step 5. Look at those equations, if any, which have nothing (zero coefficients) on the left. Remove the trivial ones, i.e. which have 0 on the right side (it might happen that after all the eliminations one of the equation looks like e.g. $0 \cdot x_3 + 0 \cdot x_4 = 0$, then just forget it). If you have an equation in the triagular form where all coefficients on the left side are zero, but the right side is not zero, then this is a contradiction, showing that the original system has no solution (e.g. if you see something like $0 \cdot x_3 + 0 \cdot x_4 = 4$). Nothing needs to be done further.

Step 6. Now you have equations in the triangular form, and each equation is actually a pivot equation for one of the unknowns (and each unknown has at most one pivot equation). This unknown is always the first unknown (with nonzero coefficient) in the corresponding equation.

But there could be certain variables, which do not belong to any pivot equation. This occurs exactly when you wanted to eliminate x_k, say, but simply there was no x_k in the remaining equations (the pivot equations for $x_1, x_2, \dots x_{k-1}$ do not count, they might have x_k, but we don't want to touch those equations again). All these variables, which have no pivot equation, will be free variables (also called parameters), i.e. their values can be arbitrary. The solution of the whole system will be given in terms of the free variables. The nonfree (pivot) variables can be uniquely expressed by the free variables by solving the remaining equations. Of course you start solving the last equation and go upward.

Here is a picture. After having done the elimination procedure honestly and removed the all zero equations, you end up with something like, for example:

$$\begin{array}{ccccccccccccccc} x_1 & + & x_2 & + & 2x_3 & + & x_4 & - & x_5 & + & 3x_6 & - & x_7 & = & 2 \\ & & & & \frac{1}{2}x_3 & + & 2x_4 & - & 5x_5 & + & x_6 & - & 2x_7 & = & 4 \\ & & & & & & & & x_5 & + & 2x_6 & + & x_7 & = & 0 \end{array}$$

Here the first equation is pivot for x_1, the second is pivot for x_3 and the last is pivot for x_5. The variables x_2, x_4, x_6, x_7 do not belong to any pivot equation. Hence they are free parameters. The last equation is solved for its pivot variable, i.e. for x_5, you get

$$x_5 = -2x_6 - x_7.$$

The second equation is solved for x_3 and it is $x_3 = 2\Big(4 + 2x_7 - x_6 + 5x_5 - 2x_4\Big)$. This is not the final expression since it contains x_5 which is not a free parameter (in the final solution everything has to be expressed in terms of the free parameters, in this case in terms of x_2, x_4, x_6, x_7). But x_5 has been expressed in terms of x_6, x_7, so plug it back, you get

$$\begin{aligned} x_3 &= 2\Big(4 + 2x_7 - x_6 + 5(-2x_6 - x_7) - 2x_4\Big) \\ &= 8 - 4x_4 - 22x_6 - 6x_7 \end{aligned}$$

(Always simplify algebraic expressions!!!, and it is a good idea to write the free parameters in the order given by their indices).

Finally, express x_1 from the first equation, and use the above expressions for x_3 and x_5 to have a formula containing only x_2, x_4, x_6, x_7:

$$\begin{aligned} x_1 &= 2 - x_2 - 2x_3 - x_4 + x_5 - 3x_6 + x_7 \\ &= 2 - x_2 - 2\Big(8 - 4x_4 - 22x_6 - 6x_7\Big) - x_4 + \Big(-2x_6 - x_7\Big) - 3x_6 + x_7 \\ &= -14 - x_2 + 7x_4 + 39x_6 + 12x_7 \end{aligned}$$

Hence the final solution is

$$\begin{aligned} x_1 &= -14 - x_2 + 7x_4 + 39x_6 + 12x_7 \\ x_2 &= x_2 \\ x_3 &= 8 - 4x_4 - 22x_6 - 6x_7 \\ x_4 &= x_4 \\ x_5 &= -2x_6 - x_7 \\ x_6 &= x_6 \\ x_7 &= x_7 \end{aligned}$$

The funnily looking equations $x_2 = x_2$, $x_4 = x_4$ etc. express the fact that these variables are free, they can be anything, independently of each other, while x_1, x_3, x_5 are determined once the free variables are fixed.

8.2 Recording the steps (Section 1.3 of [D])

Notice that the right hand side of the system of equation determines all the steps in the Gauss elimination; the numbers on the right play no role, except that they change according to the operations.

Suppose you have to solve several systems of equations whose left sides are the same, just the numbers on the right change. Then it is advisable to record somehow the exact steps you have done during the elimination when you solved the first system. Clearly, when you solve the second system, the same steps are needed, and the left side will be exactly the same (you get the same triangular form, so you don't have to do it again), except there are new numbers on the right. But if you recorded the steps during the solution of first system, then you can perform these steps on the numbers only when you solve the second system. This saves a lot of work in the first part of the Gauss elimination. The second part (solution of the triangular system) depends heavily on the numbers on the right side, so this part you have to redo.

This is the basic idea. Now there are various formalisms to record the steps. I think the one suggested in the book is quite messy (though nothing wrong about it). So I suggest to use a simpler, more pedestrian recipe to note the steps (eventually, if you implement it on a computer, the best formalism depends on the way how your original coefficients are stored etc.).

There are three types of operations; interchanging equations, multiplying equations (with a nonzero number), and subtracting one equation from the other. In fact the last two steps are always used in a combined way: you subtract a certain multiple of an equation (typically the pivot equation) from another equation.

We can record these steps as follows. Interchanging the second and fourth equation, say, can be recorded as $(second) \longleftrightarrow (fourth)$ or $(2) \longleftrightarrow (4)$. Subtracting say four times the third equation from the fifth can be written as $(fifth) \to (fifth) - 4\cdot(third)$, or $(5) \to (5) - 4\cdot(3)$, indicating that the fifth equation is to be replaced by the fifth minus four times the third etc.

If you look at the example above, when started from

$$\begin{array}{rcrcrcr} 2x_1 & + & 3x_2 & - & x_3 & = & 3 \\ x_1 & + & x_2 & + & x_3 & = & 2 \\ x_1 & - & x_2 & & & = & 4 \end{array}$$

and ended up with its triangular form

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 2 \\ & & x_2 & - & 3x_3 & = & -1 \\ & & & - & 7x_3 & = & 0 \end{array}$$

then you can record the performed operations as:

Step 1: $(1) \longleftrightarrow (2)$

Step 2: $(2) \to (2) - 2 \cdot (1)$

Step 3: $(3) \to (3) - (1)$

Step 4: $(3) \to (3) + 2 \cdot (2)$

Why is this useful? Suppose now you have to solve the following system:

$$\begin{array}{rcrcrcr} 2x_1 & + & 3x_2 & - & x_3 & = & 5 \\ x_1 & + & x_2 & + & x_3 & = & 1 \\ x_1 & - & x_2 & + & x_3 & = & 3 \end{array}$$

Notice that the left side is the same as before, just the numbers on the right changed. So the elimination will give the same triangular form:

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & ? \\ & & x_2 & - & 3x_3 & = & ? \\ & & & - & 7x_3 & = & ? \end{array}$$

except that the right side numbers will be different. How to find them? Just perform the prescribed sequence of operations on this column of numbers:

$$\begin{array}{ccccccccc} 5 & & 1 & & 1 & & 1 & & 1 \\ 1 & \to & 5 & \to & 3 & \to & 3 & \to & 3 \\ 3 & & 3 & & 3 & & 2 & & 8 \end{array}$$

In the first step we interchanged first and second, then we subtracted twice the first from the second (i.e. $3 = 5 - 2 \cdot 1$), etc.

Hence without redoing the full Gaussian elimination, we can write up the new triangular system:

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 1 \\ & & x_2 & - & 3x_3 & = & 3 \\ & & & - & 7x_3 & = & 8 \end{array}$$

We still have to solve this triangular system, and now we cannot use the previous knowledge any more. So this step has to be redone completely. The result is $x_3 = -\frac{8}{7}$ from the last, $x_2 = -\frac{3}{7}$ from the second and finally $x_1 = \frac{18}{7}$ from the first.