9 Appendix. Determinants and Cramer's formula

Here we the definition of the determinant in the general case and summarize some features. Then we show how to use them to solve a regular linear system $A\mathbf{x} = \mathbf{b}$. This method is called the Cramer's rule, and it is not used in practice for large systems (requires too many steps and very unstable). However for theory, it is sometimes important that we have an "explicit" solution to the equation $A\mathbf{x} = \mathbf{b}$ instead of algorithms.

Never forget that the determinant is a NUMBER assigned to a square matrix. Rectangular matrices do NOT have determinants.

9.1 Definition of the determinant for $n \times n$ matrices

An n by n matrix is a square array of numbers

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

Its determinant, denoted by

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

is defined inductively. This means that in order to compute an $n \times n$ determinant, first you have to know how to compute an $(n-1) \times (n-1)$ determinant, but for that you have to know how to compute an $(n-2) \times (n-2)$ determinant, etc. until you get down to 1×1 determinant, which is just the only entry of the matrix.

n=1 case:

The determinant of an 1×1 "matrix", $A = (a)$ is just this number:

$$\det A = \det\,(a) = a$$

<u>*n=2 case:*</u>

The determinant of a 2×2 matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is given

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

(product of elements in the main diagonal minus product of elements in the off-diagonal).

Now view it in the following (stupid, but useful) way: Take the first column, and expand the determinant along this column. It means that take the first element of the first column (a_{11}), remove the column and the row of this element and consider the remaining 1×1 matrix (namely (a_{22})). Take a_{11} times the determinant of this smaller matrix. I.e. you should view $a_{11}a_{22}$ as $a_{11} \cdot \det\,(a_{22})$.

Then take the second element of the column we chose (first column); this is a_{21}, and consider the matrix obtained from the original big matrix by removing the column and row of this element. In this case this is the 1×1 matrix (a_{12}). Take its determinant, multiply by a_{21}. I.e. view $a_{12}a_{21}$ as $a_{21} \cdot \det(a_{12})$.

Now add up these two numbers (i.e. $a_{11} \cdot \det\,(a_{22})$ and $a_{21} \cdot \det(a_{12})$), but with alternating signs: the first one comes with plus, the second one with minus. This gives

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11} \cdot \det\,(a_{22}) - a_{21} \cdot \det(a_{12})$$

This sounds unnecessarily complicated, but the same procedure works for any matrix. Let's check $n = 3$:

<u>*n=3 case:*</u>

I told you in class the rule to remember a 3×3 determinant:

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} - a_{31}a_{22}a_{13}$$

I.e. take the three products obtained by reading the elements paralell with the main diagonal with plus sign and subtract from this the three products obtained by reading the elements parallel with the off-diagonal (sometimes this is called **Sarrus rule**) or **Arrow rule**.

Now we can obtain the same result by "expansion".

Choose the first column. Take the first element a_{11}, and consider the matrix obtained by removing the first column and first row from the original matrix (i.e. the column and row of the chosen element). This is a 2×2 matrix, take its determinant and multiply it by a_{11}. Hence we have from a_{11}:

$$a_{11} \cdot \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{11}\Big(a_{22}a_{33} - a_{23}a_{32}\Big)$$

Now take the second element of the first column, a_{21}. Consider the 2×2 matrix after removing the row and column of a_{21}, compute its determinant and multiply by a_{21}:

$$a_{21} \cdot \det \begin{pmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{pmatrix} = a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{21}\Big(a_{12}a_{33} - a_{13}a_{32}\Big)$$

But this quantity must come with a MINUS sign in the final formula.

Finally, take the third element of the first column, a_{31}, remove its row and column, compute the determinant of the 2×2 matrix:

$$a_{31} \cdot \det \begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix} = a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = a_{31}\Big(a_{12}a_{23} - a_{13}a_{22}\Big)$$

This number comes with plus sign.

Hence the final formula:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11}\Big(a_{22}a_{33} - a_{23}a_{32}\Big) - a_{21}\Big(a_{12}a_{33} - a_{13}a_{32}\Big) + a_{31}\Big(a_{12}a_{23} - a_{13}a_{22}\Big)$$

which coincides with the result obtained by the Sarrus rule.

General case:

WARNING: The Sarrus rule (product of elements in the direction of the diagonal minus the product of elements in the offdiagonal direction) DOES NOT APPLY in general (only for $n = 2, 3$). The general definition of the determinant is via expansion, and here it is:

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & \dots & a_{2n} \\ a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & & \vdots \\ a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix}$$

$$-a_{21} \begin{vmatrix} a_{12} & a_{13} & \dots & a_{1n} \\ a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & & \vdots \\ a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} & \dots & a_{1n} \\ a_{22} & a_{23} & \dots & a_{2n} \\ a_{42} & a_{43} & \dots & a_{4n} \\ \vdots & \vdots & & \vdots \\ a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix} - \dots$$

In words: we expand along the first column. Take the first element a_{11}, and multiply it with the determinant of the $(n-1) \times (n-1)$ matrix obtained from the original by removing the column and the row of a_{11}. This quantity comes with a PLUS sign. Then take the second element of the first column (according to which you expand), this is a_{21}, and multiply this by the determinant of the $(n-1) \times (n-1)$ matrix obtained from the original by removing the column and the row of a_{21}. This quantity comes with a MINUS sign. Then take the third element a_{31} and multiply this by the determinant of the $(n-1) \times (n-1)$ matrix obtained from the original by removing the column and the row of a_{31}. This quantity comes with a PLUS sign. etc.

Remark: You can expand the determinant along any other column, not necessarily the first. The signs are always alternating, but whether you start with PLUS or MINUS, depends on the column according which you expand. I.e. if you expand according to the second column, then the product of a_{12} times the determinant of the corresponding $(n-1) \times (n-1)$ matrix comes with a MINUS sign, then a_{22} times the corresponding $(n-1) \times (n-1)$ matrix comes with a PLUS sign, etc. I.e. the order of signs is MINUS, PLUS, MINUS, etc., unlike the PLUS, MINUS, PLUS etc. in case of the expansion along the first column. The expansion along the odd columns (first, third, fifth etc.) follows the pattern of the first column (PLUS, MINUS, PLUS, etc.), the expansion along the even columns (second, fourth, sixth etc.) follows the patter MINUS, PLUS, MINUS, etc.

This formula tells you how to reduce the computation of an n by n determinant to the computation of n determinants of size $n-1$ by $n-1$ (Again: select any column - the first one in the example above -, and multiply each element by the determinant of the corresponding submatrix, and add them up with an alternating sign. The submatrix of any element is obtained by deleting the full row and column of that element.)

This is a proper definition of the determinant, but not very useful for computations.

To compute a big determinant, you should use the following rules to reduce it to an upper triangular form. A matrix is called upper triangular, if all the elements below the main diagonal is zero:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1,n-1} & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2,n-1} & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3,n-1} & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 & a_{nn} \end{pmatrix}$$

The determinant of such a matrix is the product of the diagonal elements:

$$\det A = a_{11}a_{22}a_{33}\dots a_{n-1,n-1}a_{nn}$$

Now, using the following steps, you can always bring the determinant into an upper triangular form. You can easily notice that these steps are essentially the same steps we used in Gaussian elimination:

STEPS TO COMPUTE DETERMINANTS:

1, The value of any determinant with a row full of zeros is zero.

2, If we multiply each element in a single row by a constant, then the value of the determinant is multiplied by the same constant.

3, The value of the determinant does not change if we add a constant multiple of any of its row to another row.

4, The value of the determinant changes to its negative if we interchange any two rows.

5, 1,-4, hold true if we replace "row" by "column".

There is one more rule, which you can use, but not necessary, and which has no counterpart in Gaussian elimination (this is something which makes sense only for square matrices):

6, The value of the determinant does not change if we reflect it on the main diagonal.

WARNING: The determinant function is not additive, in other words

$$\det(A + B) \neq \det(A) + \det(B)$$

However it happens to be multiplicative

$$\det(AB) = \det(A)\det(B)$$

but we do not prove this fact here.

9.2 Solution to a linear system using determinants: Cramer's rule

Suppose you want to solve the following system of n equations and n unknowns (it is important that the number of equations is equal to the number of unknowns, this gives square matrix)

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n$$

In other words, you want to solve $Ax = b$ where

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ a_{21} & \ldots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Then the system has a unique solution for any b if and only if $\det A \neq 0$. In this case $R(A) = \mathbf{R}^n$ and $N(A) = \{\mathbf{0}\}$. This is the nicest "typical" case for n equations and n unknowns.

Suppose now that $\det A \neq 0$. Then for any b, the solution is the following:

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & \dots & a_{1n} \\ b_2 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \dots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}} \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & \dots & a_{1n} \\ a_{21} & b_2 & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & b_n & \dots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}} \qquad \dots$$

etc.

$$x_n = \frac{\begin{vmatrix} a_{11} & a_{12} & \dots & b_1 \\ a_{21} & a_{22} & \dots & b_2 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{2n} & \dots & b_n \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}}$$

i.e. the j-th unknown x_j is a ratio of two determinants. The denominator is always $\det A$; the numerator is almost the same, except that you replace the j-th column of A by the column vector b.

This formula is called **Cramer's rule**.