I decided to grade the following problems.

Section 12.3 43)

Let $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ be nonzero vectors such that $\|\vec{\mathbf{a}}-\vec{\mathbf{b}}\| = \|\vec{\mathbf{a}}+\vec{\mathbf{b}}\|$.

(a) What can you conclude about the parallelogram generated by $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$?

(b) Show that if $\vec{\mathbf{a}} = (a_1, a_2, a_3)$ and $\vec{\mathbf{b}} = (b_1, b_2, b_3)$, then $a_1b_1 + a_2b_2 + a_3b_3 = 0$.

__**Solution**__:

(a) The two quantities $\|\vec{\mathbf{a}}-\vec{\mathbf{b}}\|$ and $\|\vec{\mathbf{a}}+\vec{\mathbf{b}}\|$ denote the magnitude of the diagonals for the parallelogram. The only time these diagonals can have the same length is if the parallelogram is a rectangle.

(b) Some people noticed that this relation is exactly the dot product of the vector $\vec{\mathbf{a}}$ with $\vec{\mathbf{b}}$. I didn't take off points for the dot product reasoning but since we don't really know about dot products yet, our only hope is to fiddle with the information we do have. So,

$$\|\vec{\mathbf{a}}-\vec{\mathbf{b}}\| = \|\vec{\mathbf{a}}+\vec{\mathbf{b}}\| \Rightarrow \|(a_1-b_1, a_2-b_2, a_3-b_3)\| = \|(a_1+b_1, a_2+b_2, a_3+b_3)\|$$

$$\Rightarrow \sqrt{(a_1-b_1)^2 + (a_2-b_2)^2 + (a_3-b_3)^2} = \sqrt{(a_1+b_1)^2 + (a_2+b_2)^2 + (a_3+b_3)^2}$$

$$\Rightarrow (a_1-b_1)^2 + (a_2-b_2)^2 + (a_3-b_3)^2 = (a_1+b_1)^2 + (a_2+b_2)^2 + (a_3+b_3)^2$$

$$\Rightarrow a_1^2 - 2a_1b_1 + b_1^2 + a_2^2 - 2a_2b_2 + b_2^2 + a_3^2 - 2a_3b_3 + b_3^2 = (a_1+b_1)^2 + (a_2+b_2)^2 + (a_3+b_3)^2$$

$$\Rightarrow 0 = 4(a_1b_1 + a_2b_2 + a_3b_3)$$

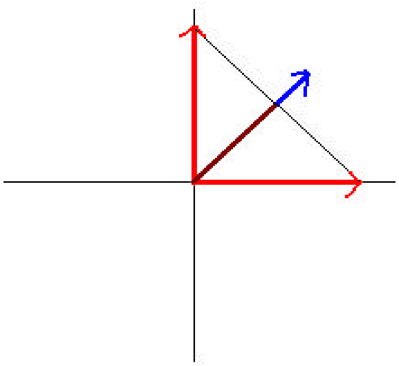
$$\Rightarrow 0 = a_1b_1 + a_2b_2 + a_3b_3 \text{ as desired.}$$

Section 12.4 29)

(a) (Important) Let $\vec{\mathbf{a}} \neq \vec{\mathbf{0}}$. Show that $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{c}}$ does not necessarily imply that $\vec{\mathbf{b}} = \vec{\mathbf{c}}$, but only that $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ have the same projection on $\vec{\mathbf{a}}$. Draw a figure illustrating this for $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ different from $\vec{\mathbf{0}}$.

(b) Show that if for all unit vectors $\vec{\mathbf{u}} \cdot \vec{\mathbf{b}} = \vec{\mathbf{u}} \cdot \vec{\mathbf{c}}$, then $\vec{\mathbf{b}} = \vec{\mathbf{c}}$. HINT: Consider the unit coordinate vectors.

<u>Solution</u>:



(a) To show that the implication does not always hold consider $\vec{\mathbf{a}} = (1,1)$, $\vec{\mathbf{b}} = \vec{e}_1 = (1,0)$, and $\vec{\mathbf{c}} = \vec{e}_2 = (0,1)$. Then we have the following computation $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = (1,1) \cdot (1,0) = 1 = (1,1) \cdot (0,1) = \vec{\mathbf{a}} \cdot \vec{\mathbf{c}}$. So, $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{c}}$ but $\vec{\mathbf{b}} \neq \vec{\mathbf{c}}$. We can see this in the figure to the left.

To see that $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ have the same projection on $\vec{\mathbf{a}}$ we perform the following algebra calculation.

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{c}}$$

$$\Leftrightarrow \vec{\mathbf{b}} \cdot \vec{\mathbf{a}} = \vec{\mathbf{c}} \cdot \vec{\mathbf{a}}$$

$$\Leftrightarrow \left(\vec{\mathbf{b}} \cdot \vec{\mathbf{a}}\right)\|\vec{\mathbf{a}}\|^{-2} = \left(\vec{\mathbf{c}} \cdot \vec{\mathbf{a}}\right)\|\vec{\mathbf{a}}\|^{-2}$$

$$\Leftrightarrow \left(\vec{\mathbf{b}} \cdot \frac{\vec{\mathbf{a}}}{\|\vec{\mathbf{a}}\|}\right)\frac{\vec{\mathbf{a}}}{\|\vec{\mathbf{a}}\|} = \left(\vec{\mathbf{c}} \cdot \frac{\vec{\mathbf{a}}}{\|\vec{\mathbf{a}}\|}\right)\frac{\vec{\mathbf{a}}}{\|\vec{\mathbf{a}}\|}$$

which by definition gives us $proj_{\vec{\mathbf{a}}}\vec{\mathbf{b}} = proj_{\vec{\mathbf{a}}}\vec{\mathbf{c}}$.

(b) A lot of you had the right idea on this problem, but you need to be a little more thorough in your explanation. Let $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ be vectors in $\mathbb{R}^n$ so that $\vec{\mathbf{u}} \cdot \vec{\mathbf{b}} = \vec{\mathbf{u}} \cdot \vec{\mathbf{c}}$ for all unit vectors $\vec{\mathbf{u}}$. You should read this as, "the magnitude of $\vec{\mathbf{b}}$ in any direction $\vec{\mathbf{u}}$ is the same as the magnitude of $\vec{\mathbf{c}}$ in the direction $\vec{\mathbf{u}}$." Next, recall that two vectors are equal if and only if they agree in each coordinate. That is $\vec{\mathbf{b}} = (b_1, b_2, \ldots, b_n)$ is equal to $\vec{\mathbf{c}} = (c_1, c_2, \ldots, c_n)$ if and only if $b_i = c_i$ for $i = 1, 2, \ldots, n$. We need some way to pull each component out of the vectors so that we may compare them one at a time. Fortunately, taking the dot product with the i^{th} direction vector pulls out the i^{th} component. That is $\vec{\mathbf{b}} \cdot \vec{e}_1 = (b_1, b_2, \ldots, b_n) \cdot (1, 0, \ldots, 0) = b_1$, $\vec{\mathbf{b}} \cdot \vec{e}_2 = (b_1, b_2, \ldots, b_n) \cdot (0, 1, \ldots, 0) = b_2$, and in general, $\vec{\mathbf{b}} \cdot \vec{e}_i = b_i$. So, we have $\vec{\mathbf{b}} \cdot \vec{e}_i = \vec{\mathbf{c}} \cdot \vec{e}_i$ for $i = 1, 2, \ldots, n$ since each $\vec{e}_i$ is a unit vector (remember this was given to us in the statement of the problem). So, we have $b_i = \vec{\mathbf{b}} \cdot \vec{e}_i = \vec{\mathbf{c}} \cdot \vec{e}_i = c_i$ for $i = 1, 2, \ldots, n$ and so the vectors are equal. In fact, we have the following $\vec{\mathbf{b}} = \left(\left(\vec{\mathbf{b}} \cdot \vec{e}_1\right)\vec{e}_1, \left(\vec{\mathbf{b}} \cdot \vec{e}_2\right)\vec{e}_2, \ldots, \left(\vec{\mathbf{b}} \cdot \vec{e}_n\right)\vec{e}_n\right) = \left(\left(\vec{\mathbf{c}} \cdot \vec{e}_1\right)\vec{e}_1, \left(\vec{\mathbf{c}} \cdot \vec{e}_2\right)\vec{e}_2, \ldots, \left(\vec{\mathbf{c}} \cdot \vec{e}_n\right)\vec{e}_n\right) = \vec{\mathbf{c}}$.

Section 12.6 21)

Find a vector parameterization for the line segment that begins at $(2,7,-1)$ and ends at $(4,2,3)$.

Solution: We are looking for a segment of a line. This can be achieved by limiting the parameter values in the vector equation of a line. We know an arbitrary line has the form $\mathbf{r}(t)=\vec{\mathbf{r}}_0+t\vec{\mathbf{d}}$ where $\vec{\mathbf{d}}$ is a vector in the direction of the line and $\vec{\mathbf{r}}_0$ represents a vector whose tip is at some point on the line. We are looking for a piece of a line that runs through the points $(2,7,-1)$ and $(4,2,3)$. So, then we can immediately find $\vec{\mathbf{d}}=(4,2,3)-(2,7,-1)=(2,-5,4)$ since we want the line to connect these two points. Thus the line must go in the direction of a vector whose tail is at one point and the tip is at the other point. We can choose $\vec{\mathbf{r}}_0=(2,7,-1)$ and we have the equation of the line $\mathbf{r}(t)=(2,7,-1)+t(2,-5,4)$. To make sure that we only have line segment connecting the two points we only need to restrict the time parameter. Thus the desired parameterization is $\mathbf{r}(t)=(2,7,-1)+t(2,-5,4)$ where $t\in[0,1]$. Recall in your notes that I introduced vectors by saying we would like to solve a problem similar to this one. The parameterization I used in the class notes was $(1-t)(2,7,-1)+t(4,2,3)$ where $t\in[0,1]$. If you do a little bit of rearranging you will see that it is exactly the same thing.