Solutions to Homework 2

Since we solved all of the web problems in class, I have written up solutions to problems in §12.7. If you need help with a web problem please send me an email (mullikin@math.gatech.edu) and I will answer your question. Sorry I didn't get these up earlier. It took a lot longer than I expected to write all of this up.

Before I begin I need to describe my notation. Whenever you see $< x, y, z >$ it is understood to be a vector. Whenever you see (x, y, z) it is understood to be a point.

§12.7 3)

Find an equation for the plane that passes through the point $(2, 3, 4)$ and is perpendicular to $< 1, -4, 3 >$.

Answer : Since we know that the plane is perpendicular to $< 1, -4, 3 >$ then we know that a perfectly good normal vector is $\vec{N} =< 1, -4, 3 >$. Since we also have a point given to us there isn't much else to do. Let us generalize a little bit. If (x, y, z) is any point in the plane then I can find the vector whose tail is at $(2, 3, 4)$ and whose head is at (x, y, z) by pretending the two points are vectors and subtracting them. That is the vector $< x - 2, y - 3, z - 4 >$ is a vector in the plane. So, since it is a vector in the plane, it must be perpendicular to $\vec{N}$. Thus, $\vec{N} \cdot < x - 2, y - 3, z - 4 >= 0$. This is enough information to describe the plane. The desired plane is the set of all points (x, y, z) so that $\vec{N} \cdot < x - 2, y - 3, z - 4 >= 0$. Or in matheese, $\mathbb{P} = \{(x, y, z) \mid \vec{N} \cdot < x - 2, y - 3, z - 4 >= 0\} = \{(x, y, z) \mid < 1, -4, 3 > \cdot < x - 2, y - 3, z - 4 >= 0\} = \{(x, y, z) \mid (x - 2) - 4(y - 3) + 3(z - 4) = 0\}$.

§12.7 5)

Find an equation for the plane that passees through the point $(2, 1, 1)$ and is parallel to the plane $3x - 2y + 5z - 2 = 0$.

Answer : Since the deisred plane is parallel to the given plane we know that they must have parallel normal vectors. So, if we can find the normal vector for the given plane we will have a perfectly good normal vector for the desired plane. Unfortunately, the equation provided to us is in a format which does not immediately allow us to read of the normal vector. This dilemma is similar to having to complete the square on an equation to get it to look like the equation of a circle, hyperbola, ellipse, etc... The interesting equation for a plane is $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$. From this equation we can easily read off the normal vector $\vec{N} =< A, B, C >$, and we also know that (x_0, y_0, z_0) is a point in the plane. So, how can we write $3x - 2y + 5z - 2 = 0$ in this form? Well the only difference between these equations is that the one we are given has that silly '2' floating around. However, we can do a little algebra to clean things up. $3x - 2y + 5z - 2 = 3x - 2(y + 1) + 5z = 3(x - 0) - 2(y + 1) + 5(z - 0)$. All I did was pair up the constant with the y term. Why did I choose y you may ask? Because the y term had a '2' in front of it as well so it was easy to factor. I could just have well paired it up with the z or the x terms. I.e.,
$3x - 2y + 5z - 2$
$= 3(x - \frac{2}{3}) - 2y + 5z$
$= 3x - 2(y + 1) + 5z$
$= 3x - 2y + 5(z - \frac{2}{5})$
Use whichever you like. The important thing is that we can read off the normal vector $\vec{N} =< 3, -2, 5 >$. We started off with a point in the plane, so, using the same techniques from problem 3), we have $\mathbb{P} = \{(x, y, z) \mid 3(x - 2) - 2(y - 1) + 5(z - 1) = 0\}$.

§12.7 7)

Find an equation for the plane that passes through the point $(1,3,1)$ and contains the line $l : x = t, y = t, z = -2 + t$.

Answer : Again, since we already have a point on the line we only need to find the normal vector. We now must revist the awesome power of the *cross product*! I get chills. I can rewrite the equation l as the following, $l(t) = (t, t, -2+t)$, or in vector notation $\mathbf{l}(t) =< 0,0,0 > +t < 1,1,1 >$. I now have a vector in the plane. Indeed, if the entile line l lies in the plane, then its direction vector $\vec{d} =< 1,1,1 >$ is in the plane. We only need to find one more vector in the plane, which is not parallel to $\vec{d}$, and we may then use the cross product to find the normal vector. Here we need to use the other point given to us. Since I know $(1,3,1)$ is in the plane and since I know every point on the line l is in the plane I can for a vector whose tail is on any 'ol point in the line l and whose head is on the point $(1,3,1)$. This vector has the form $< 1-t, 3-t, 1-(-2+t) >$ where I can pick any value for t that I want since I only need *one* point on the line along with $(1,3,1)$ to find the desired vector. So, I'll choose $t = \frac{\sqrt{\pi^e+\sqrt{2}}}{\sqrt{\sqrt{\sqrt{75{,}321}}}}$ ... just kidding. I could use this but I think I'll stick with one of my favorites $t = 0$. So, I know have the vectors $\vec{d} =< 1,1,1 >$ and $< 1-0, 3-0, 1-(-2+0) >=< 1,3,3 >$. Thus, we have the normal vector $\vec{N} =< 1,1,1 > \times < 1,3,3 >$. So,

$$\vec{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 3 & 3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 0\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} =< 0,-2,2 >.$$

So, the equation of the plane is $\mathbb{P} = \{(x,y,z) \mid -2(y-3) + 2(z-1) = 0\}$.

§12.7 15)

Find the angle between the planes $\mathbb{P}_1 = \{(x, y, z) \mid 5(x-1)-3(y+2)+2z = 0\}$ and $\mathbb{P}_2 = \{(x, y, z) \mid x + 3(y - 1) + 2(z + 4) = 0\}$.

Answer : Recall that the angle between two planes is none other than the angle between the normal vectors $\vec{N}_1$ and $\vec{N}_2$ of the planes $\mathbb{P}_1$ and $\mathbb{P}_2$ respectively. So, fortunately, the equations given to use are already in a nice form we can read off the normal vectors $\vec{N}_1 =< 5, -3, 2 >$ and $\vec{N}_2 =< 1, 3, 2 >$. Next remember that the angle θ between the two vectors $\vec{N}_1$ and $\vec{N}_2$ is $\cos\theta = |\vec{N}_1 \cdot \vec{N}_2|$. First we must compute the unit vectors $\vec{N}_1$ and $\vec{N}_2$. $\vec{N}_1 = \frac{\vec{N}_1}{\|\vec{N}_1\|} = \frac{<5,-3,2>}{\sqrt{5^2+(-3)^2+2^2}} = \frac{1}{\sqrt{38}} < 5, -3, 2 >$. By a similar argument, $\vec{N}_2 = \frac{1}{\sqrt{14}} < 1, 3, 2 >$. So, we have $\cos\theta = |\vec{N}_1 \cdot \vec{N}_2| = |\frac{1}{\sqrt{38}} < 5, -3, 2 > \cdot \frac{1}{\sqrt{14}} < 1, 3, 2 > | = |\frac{1}{\sqrt{38}\sqrt{14}}|| < 5, -3, 2 > \cdot < 1, 3, 2 > | = \frac{1}{\sqrt{38}\sqrt{14}}|5 - 9 + 4| = 0$. Thus, $\cos\theta = 0$ tells us that $\theta = \frac{\pi}{2}$.

§12.7 21)

Determine whether the vectors $< 1, 1, 1 >$, $< 2, -1, 0 >$, and $< 3, -1, -1 >$ are coplanar.

Answer : Let us translate this problem. If these three vectors are coplanar then what does that mean in terms of linear dependenc/independence? Well, the dimension of a plane is equal to the number of elements in the minimal spanning set (i.e. the basis). So, we have the dimension of the plane is 2. Thus 2 is also the number of vectors in the maximal linearly independent set of vectors in the plane (since this is also the basis). Since the maximal linearly independent set of vectors in the plane is 2 we know that there cannont be three linearly independent vectors in the plane without violating maximality. Thus, if the three vectors $< 1, 1, 1 >$, $< 2, -1, 0 >$, and $< 3, -1, -1 >$ form a linearly dependent set, then they all must lie in the same plane. If they are linearly independent, then they form a basis for $\mathbb{R}^3$. So, we need to find α_1, α_2, and α_3 so that $\alpha_1 < 1, 1, 1 > +\alpha_2 < 2, -1, 0 > +\alpha_3 < 3, -1, -1 >= \vec{0}$. If $\alpha_1 = \alpha_2 = \alpha_3 = 0$, then the vectors are linearly independent and so they are *not* coplanar. Otherwise, they are coplanar. We have,

$\alpha_1 < 1, 1, 1 > +\alpha_2 < 2, -1, 0 > +\alpha_3 < 3, -1, -1 >= \vec{0}$

$\Rightarrow < \alpha_1 + 2\alpha_2 + 3\alpha_3, \alpha_1 - \alpha_2 - \alpha_3, \alpha_1 - \alpha_3 >=< 0, 0, 0 >$

So, we have three equations and three unknowns.

$$\begin{array}{ccccccl} \alpha_1 & + & 2\alpha_2 & + & 3\alpha_3 & = 0 \\ \alpha_1 & - & \alpha_2 & - & \alpha_3 & = 0 \\ \alpha_1 & & & - & \alpha_3 & = 0 \end{array}$$

Using the last equation, we see that $\alpha_1 = \alpha_3$. Using this new information in the second equation we see that $\alpha_2 = 0$. Finally, looking at the first equation we see that it must be the case that $\alpha_1 = \alpha_2 = \alpha_3 = 0$. The vectors are linearly independent, thus they form a maximal linearly independent set in $\mathbb{R}^3$, thus they are a basis for $\mathbb{R}^3$, thus they *can't* all lie in the same plane.

§12.7 23)

Find the distance from the point $(2, -1, 3)$ to the plane $\mathbb{P} = \{(x, y, z) \mid 2x + 4y - z + 1 = 0\}$.

Answer : We know that the distance from a point $p = (x_1, y_1, z_1)$ to a plane $\mathbb{P} = \{(x, y, z) \mid Ax + By + Cz + D = 0\}$ is given by the formula

$$d(\mathbb{P}, p) = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

To see where this come from check out the big blue calculus book pg. 765-766 and see if you can follow the proof. We also did it in class, so you may want to look in your notes. Anyhoo, we have all the information we need.

$$d(\mathbb{P}, p) = \frac{|(2)(2) + (4)(-1) + (-1)(3) + 1|}{\sqrt{2^2 + 4^2 + (-1)^2}} = \frac{2}{\sqrt{21}}$$

§12.7 29)

Find an equation in (x, y, z) for the plane that passes through the points $p_1 = (3, -4, 1)$, $p_2 = (3, 2, 1)$, and $p_3 = (-1, 1, -2)$.

Answer : We are initially given a point in the plane so all we need now is the normal vector. It would suffice to find two vectors in the plane that are not parallel. We could then take thieir cross product to find the normal vector. So, consider the vectors $\vec{p_1p_2} =< 3-3, 2-(-4), 1-1 >=< 0, 6, 0$ and $\vec{p_1p_3} =< -1-3, 1-(-4), -2-1 >=< -4, 5, -3 >$. Then, $\vec{N} = < \vec{0, 6, 0 \times < -4, 5}, -3 > =< -18, 0, 24 >$. Notice that the only part of the normal vector we care about is the direction. Thus, we could let $\vec{N} =< -3, 0, 4 >$ by dividing each component by 6. This is not necessary but it can make the computation a little easier. So, we have $\mathbb{P} = \{(x, y, z) \mid -3(x-3) + 4(z-1) = 0\}$.

§12.7 37)

Let l be the line determined by $p_1 = (1, -1, 2)$, $p_2 = (-2, 3, 1)$, and let $\mathbb{P}$ be the plane determined by $q_1 = (2, 0, -4)$, $q_2 = (1, 2, 3)$, and $q_3 = (-1, 2, 1)$. Where, if anywhere, does l intersect $\mathbb{P}$?

Answer : First, it seems like a good idea to find the equations for l and $\mathbb{P}$. We have, $\mathbf{l}(t) =< 1, -1, 2 > +t < -2 - 1, 3 + 1, 1 - 2 >=< 1 - 3t, -1 + 4t, 2 - t >$. For reasons we will see later let me write the equation of the line in paramentric form,

$$\begin{aligned} x(t) &= 1 - 3t \\ y(t) &= -1 + 4t \\ z(t) &= 2 - t \end{aligned}$$

Using the same techniques from the last problem we find the plane is given by taking the cross product of $q_1\vec{q}_2 =< -1, 2, 7 >$ with $q_1\vec{q}_3 =< -3, 2, 5 >$ to obtain the normal vector $\vec{N} =< -4, -16, 4 >$. Again, to make life easier I will use $\vec{N} =< -1, -4, 1 >$ as my normal vector. So, the equation of the plane is then $\mathbb{P} = \{(x, y, z) \mid -1(x - 2) - 4(y - 0) + 1(z - (-4))\}$. If there is any point on the line that intersects the plane, it must satisfy the equation for the plane. So, what does an arbitrary point on the line look like. Well, if p is any point on the line, then $p = (x(t), y(t), z(t))$ for some t, where $x(t)$, $y(t)$, and $z(t)$ are the parametric equations for the line we found above. So, let us set up the equation. We need to find a t so that $-1(x(t) - 2) - 4(y(t) - 0) + 1(z(t) - (-4)) = 0$
$\Rightarrow -1((1 - 3t) - 2) - 4(-1 + 4t) + 1((2 - t) + 4) = 0$
$\Rightarrow 1 + 3t + 4 - 16t + 6 - t = 0$
$\Rightarrow -14t + 11 = 0$
$\Rightarrow t = \frac{11}{14}$
From here you should check that the point $(1 - 3(\frac{11}{14}), -1 + 4\frac{11}{14}, 2 - \frac{11}{14})$ does indeed satisfy the equation for the plane. If it does not then you know you have made an arithmetic error.