Math 2601 C2
Homework 3

Since we will not have covered any new material until Wednesday I will only assign three problems. These involve techniques from [Notes:TH] sections 3 and 4. Please do all three and email me if you need any assistance (mullikin@math.gatech.edu). They are to be turned in Friday Jan 26, 2001 at 2:05pm. ***Homework is to be stapled (if more than one page) and solutions are to be neatly written.*** If I can't read your work, I can't give you any credit.

Problem 1 Solve the following system of equations, if possible. Are there any solutions? If so, how many (one or infinitely many)?

$$\begin{aligned} 2x + 3y - 2z &= 1 \\ -2y + 4z &= 0 \\ x + 2y - 4z &= 3 \end{aligned}$$

Solution : To solve this problem we only need to use row reduction techniques on the matrix associated with the given equations. So,

$$\begin{bmatrix} 2 & 3 & -2 & 1 \\ 0 & -2 & 4 & 0 \\ 1 & 2 & -4 & 3 \end{bmatrix} \quad (R_3 = R3 + (-\tfrac{1}{2})R_1) \begin{bmatrix} 2 & 3 & -2 & 1 \\ 0 & -2 & 4 & 0 \\ 0 & \frac{1}{2} & -3 & \frac{5}{2} \end{bmatrix}$$

$$(-\tfrac{1}{2}R_2) \begin{bmatrix} 2 & 3 & -2 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & \frac{1}{2} & -3 & \frac{5}{2} \end{bmatrix} \quad (R_3 = R3 + (-\tfrac{1}{2}R_2) \begin{bmatrix} 2 & 3 & -2 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -2 & \frac{5}{2} \end{bmatrix}$$

$$(-\tfrac{1}{2}R_3) \begin{bmatrix} 2 & 3 & -2 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -\frac{5}{4} \end{bmatrix} \quad (R_2 = R2 + (2)R_3) \begin{bmatrix} 2 & 3 & -2 & 1 \\ 0 & 1 & 0 & -\frac{5}{2} \\ 0 & 0 & 1 & -\frac{5}{4} \end{bmatrix}$$

$$(R1 = R_1 + (-3)R_2) \begin{bmatrix} 2 & 0 & -2 & \frac{17}{2} \\ 0 & 1 & 0 & -\frac{5}{2} \\ 0 & 0 & 1 & -\frac{5}{4} \end{bmatrix} \quad (R_1 = R1 + (2)R_3) \begin{bmatrix} 2 & 0 & 0 & 6 \\ 0 & 1 & 0 & -\frac{5}{2} \\ 1 & 0 & 1 & -\frac{5}{4} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -\frac{5}{2} \\ 0 & 0 & 1 & -\frac{5}{4} \end{bmatrix} \quad \begin{aligned} x &= 3 \\ y &= -\tfrac{5}{2} \\ z &= -\tfrac{5}{4} \end{aligned}$$

So we can see that there is only one solution.

Problem 2 Find $Ker(A)$ where,

$$A = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

Solution : To find $Ker(A)$ we are to find the set $\{\vec{x} \mid A\vec{x} = 0\}$. Thus we only need to perform row reduction techniques again.

$$\begin{bmatrix} 1 & 3 & 2 & 0 \\ -1 & 5 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{bmatrix} \qquad (R_2 = R_2 + R_1) \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 8 & 4 & 0 \\ 2 & 2 & 2 & 0 \end{bmatrix}$$

$$(R_3 = R_3 + (-2)R_1) \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 8 & 4 & 0 \\ 0 & -4 & -2 & 0 \end{bmatrix} \qquad (R_3 = R_3 + \tfrac{1}{2}R_1) \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 8 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(\tfrac{1}{8}R_2) \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (R_1 = R_1 + (-3)R_1) \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus we see that any vector of the form $< -\frac{1}{2}t, -\frac{1}{2}t, t >$ is in $Ker(A)$. I.e. $Ker(A) = \textbf{span}\{< -\frac{1}{2}, -\frac{1}{2}, 1 >\}$.

Problem 3 Find a scalar λ and a vector $\vec{x}$ so that $B\vec{x} = \lambda\vec{x}$ where,

$$B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Solution : Here we have a really interesting dilemma. Without knowing anything about eigenvalues or eigenvectors how can we find solutions to this problem? Well, just try to solve the given system $B\vec{x} = \lambda\vec{x}$. How are we going to do this? Go ahead say it. That's right *row reduction*. We are looking to solve the system,

$$\begin{aligned} x_1 + 2x_2 &= \lambda x_1 \\ 2x_1 + x_2 &= \lambda x_2 \end{aligned}$$

Which presents us with the matrix,

$$\begin{bmatrix} (1-\lambda) & 2 & 0 \\ 2 & (1-\lambda) & 0 \end{bmatrix}$$

Bring on the row reduction.

$$\begin{bmatrix} (1-\lambda) & 2 & 0 \\ 2 & (1-\lambda) & 0 \end{bmatrix} \quad \begin{matrix} (R_2 = R_2 + -\frac{2}{1-\lambda}) \\ (\text{So, } \lambda \neq 1) \end{matrix} \begin{bmatrix} (1-\lambda) & 2 & 0 \\ 0 & \frac{(1-\lambda)^2-4}{1-\lambda} & 0 \end{bmatrix}$$

Let's pause for a moment and reflect on recent events. Notice the last line in the matrix has a nasty looking term. For this system to have a unique solution requires that $\frac{(1-\lambda)^2-4}{1-\lambda} \neq 0$, otherwise the last row is all zero and we would have infinitely many solutions. I think you can see that if $\frac{(1-\lambda)^2-4}{1-\lambda} \neq 0$ then after performing some more row reduction we can reduce the matrix so that the solutions are $x_1 = 0$ and $x_2 = 0$. According to the question I asked, this is a valid solution. That is $B\vec{0} = \lambda\vec{0}$ for any 'ol λ you would like to choose. This case is somewhat unsatisfying. We would like to obtain some nonzero solution. How can we guarantee a nonzero solution? Well if the last row were all zero then we would have infinitely many solutions right? So, lets try that. For the last row to be all zero we only need to find out when $\frac{(1-\lambda)^2-4}{1-\lambda} = 0$. Well, $\frac{(1-\lambda)^2-4}{1-\lambda} = 0$ when $(1-\lambda)^2 - 4 = 0$. So, after multiplying everything out we see we want to solve the quadratic equation $\lambda^2 - 2\lambda - 3 = 0$. Thus we see that $\lambda = 3$ or $\lambda = -1$ will satisfy this equation (just use the quadratic formula). Sticking the value $\lambda = 3$ into the system of equations,

$$\begin{aligned} x_1 + 2x_2 &= \lambda x_1 \\ 2x_1 + x_2 &= \lambda x_2 \end{aligned}$$

we see that we obtain the solution $x_1 = x_2$. Sure enough,

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_1 \end{bmatrix} = 3 \begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

Likewise, if we use $\lambda = -1$ in the system of equations we can find that we obtain $x_1 = -x_2$. Notice, as if by magic, we have

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} = \begin{bmatrix} -x_1 \\ x_1 \end{bmatrix} = -1 \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

What in the world is going on here. Let's examine the situation a little more closely. We wanted to find a vector $\vec{x}$ and a scalar λ so that $B\vec{x} = \lambda\vec{x}$. Or, equivalently, we wanted to find a solution to $(B - \lambda I)\vec{x} = \vec{0}$ right? So, we want to find $Ker(B - \lambda I)$. So, you may recall from Calc II, that the kernel of a matrix is nontrivial (i.e. something other than the zero vector) if the determinant is zero. So, if we want to have nontrivial elements in $Ker(B - \lambda I)$ then it suffices to find a λ so that $det(B - \lambda I) = 0$. In our case, what is $det(B - \lambda I)$? It is the following,

$$det(B - \lambda I) = det(\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = det \begin{bmatrix} (1-\lambda) & 2 \\ 2 & (1-\lambda) \end{bmatrix}$$
$$= (1 - \lambda)^2 - 4$$

Does this look familiar at all? You bet it does, it's the same quadratic we had to solve earlier. Then, once you have found a value λ which makes $Ker(B - \lambda I)$ have nontrivial kernel, you can then find the 'eigenvector' (whatever that is) which is the basis for $Ker(B - \lambda I)$ for that particular λ!

Problem that will keep you up at night : You do not need to work this problem if you don't want to. But it is interesting. Why is it that there does ***not*** exist any nonzero vector $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ and a scalar $\lambda \in \mathbb{R}$ so that

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}?$$

Solution : Whuh! What's this all about? Well, what does it mean to say there exist $\vec{x}$ and λ so that $A\vec{x} = \lambda\vec{x}$? It means that $A\vec{x}$ is parallel to $\vec{x}$. Or that A only stretches or shrinks vectors in the direction of $\vec{x}$. So, why doesn't the matrix $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ have any such vectors? Well, as you may recall (or may not, that's OK we'll see it again) the above matrix is actually the matrix $\frac{2}{\sqrt{2}} \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix}$, which is the matrix of a rotation. So, since we are rotating every vector by $\frac{\pi}{4}$ radians, there is no vector which gets stretched in its original direction. Thus, there are no (nonzero) vectors which satisfy $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \vec{x} = \lambda\vec{x}$.