Math 2601 C2
Homework 4

Please do all three and email me if you need any assistance (mullikin@math.gatech.edu). They are to be turned in Friday Feb 2, 2001 at 2:05pm. ***HOMEWORK IS TO BE STAPLED AND SOLUTIONS ARE TO BE NEATLY WRITTEN.*** If I can't read your work, I can't give you any credit.

Problem 1 Let $\mathcal{V}$ be the space of all 2×2 real valued matrices. I.e.,

$$\mathcal{V} = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} | a, b, c, d \in \mathbb{R} \}$$

i) Find a basis for $\mathcal{V}$.
ii) Can you construct a linear transformation $T : \mathcal{V} \to \mathbb{R}^4$ where $Im(T) = \mathbb{R}^4$ and $Ker(T) = \vec{0}$ (Note, in this case $\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$)? Such a map is called a *bijection*, and we say $\mathcal{V}$ and $\mathbb{R}^4$ are in one-to-one correspondence.

Solution : To find a basis for V is to find a set of elements from V whose linear combinations span V and so that the elements are linearly independent. Let $\mathfrak{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. This is the standard basis for V. A perfectly good linear transformation is $\Psi_{\mathfrak{B}}$, since $\Psi_{\mathfrak{B}} \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4$.

It is 'clear' that if a matrix gets sent to zero, then all of its components must be zero as desired. It remains to show that this map is linear.

$$\Psi_{\mathfrak{B}} \left(\alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \beta \begin{bmatrix} e & f \\ g & h \end{bmatrix} \right) = \Psi_{\mathfrak{B}} \left(\begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix} + \beta \begin{bmatrix} \beta e & \beta f \\ \beta g & \beta h \end{bmatrix} \right)$$

$$= \Psi_{\mathfrak{B}} \left(\begin{bmatrix} \alpha a + \beta e & \alpha b + \beta f \\ \alpha c + \beta g & \alpha d + \beta h \end{bmatrix} \right)$$

$$= \begin{pmatrix} \alpha a + \beta e \\ \alpha b + \beta f \\ \alpha c + \beta g \\ \alpha d + \beta h \end{pmatrix} = \begin{pmatrix} \alpha a \\ \alpha b \\ \alpha c \\ \alpha d \end{pmatrix} + \begin{pmatrix} \beta e \\ \beta f \\ \beta g \\ \beta f \end{pmatrix} = \alpha \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} + \beta \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix}$$

$$= \alpha \Psi_{\mathfrak{b}} \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) + \beta \Psi_{\mathfrak{B}} \left(\begin{bmatrix} e & f \\ g & h \end{bmatrix} \right)$$

Thus we have proven that the map $\Psi_{\mathfrak{B}} : V \longrightarrow \mathbb{R}^4$ is linear and bijective.

Problem 2 Let $\mathfrak{B}_1 = \{1, x, x^2\}$ be a basis for $\mathcal{P}_2$ and let $\mathfrak{B}_2 = \{1, x, x^2, x^3\}$ be a basis for $\mathcal{P}_3$. Let $T : \mathcal{P}_2 \to \mathcal{P}_3$ be the map defined by $T(p(x)) = xp(x)$.
i) Show that T is a linear transformation by using the definition.
ii) Find the matrix ${}_{\{\mathfrak{B}_2\}}M^T_{\{\mathfrak{B}_1\}}$.
iii) Find $Ker({}_{\{\mathfrak{B}_2\}}M^T_{\{\mathfrak{B}_1\}})$ and $Im({}_{\{\mathfrak{B}_2\}}M^T_{\{\mathfrak{B}_1\}})$.

Solution :

i) We are to show that $T(\alpha p(x) + \beta q(x)) = \alpha T(p(x)) + \beta T(q(x))$ for all $p(x), q(x) \in \mathcal{P}_2$ and scalars α and β. So, let $p(x) = a_0 + a_1 x + a_2 x^2$ and let $q(x) = b_0 + b_1 x + b_2 x^2$. Then we have the following,

$$T(\alpha p(x) + \beta q(x)) = T(\alpha(a_0 + a_1x + a_2x^2) + \beta(b_0 + b_1x + b_2x^2)$$

$$= T(\alpha a_0 + \alpha a_1 x + \alpha a_2 x^2 + \beta b_0 + \beta b_1 x + \beta b_2 x^2)$$
$$= T((\alpha a_0 + \beta b_0) + (\alpha a_1 + \beta b_1)x + (\alpha a_2 + \beta b_2)x^2)$$
$$= \left[(\alpha a_0 + \beta b_0) + (\alpha a_1 + \beta b_1)x + (\alpha a_2 + \beta b_2)x^2\right] x$$
$$= (\alpha a_0 + \beta b_0)x + (\alpha a_1 + \beta b_1)x^2 + (\alpha a_2 + \beta b_2)x^3$$
$$= (\alpha a_0 x + \alpha a_1 x^2 + \alpha a_2 x^3) + (\beta b_0 x + \beta b_1 x^2 + \beta b_2 x^3)$$
$$= \alpha(a_0 + a_1x + a_2x^2)x + \beta(b_0 + b_1x + b_2x^2)$$
$$= \alpha T(a_0 + a_1x + a_2x^2) + \beta T(b_0 + b_1x + b_2x^2)$$
$$= \alpha T(p(x)) + \beta T(q(x))$$

Thus, we have shown that T is indeed linear.

ii) We know that the desired matrix has the form,

$${}_{\{\mathfrak{B}_2\}}M^T_{\{\mathfrak{B}_1\}} = \begin{bmatrix} | & | & | \\ \Psi_{\mathfrak{B}_2}(T(1)) & \Psi_{\mathfrak{B}_2}(T(x)) & \Psi_{\mathfrak{B}_2}(T(x^2)) \\ | & | & | \end{bmatrix}$$

So, we have,

$$\Psi_{\mathfrak{B}_2}(T(1)) = \Psi_{\mathfrak{B}_2}(x) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Psi_{\mathfrak{B}_2}(T(x)) = \Psi_{\mathfrak{B}_2}(x^2) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\Psi_{\mathfrak{B}_2}(T(x^2)) = \Psi_{\mathfrak{B}_2}(x^3) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Thus we have,

$$_{\{\mathfrak{B}_2\}}M^T_{\{\mathfrak{B}_1\}} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

iii) We wish first to find $Ker_{\{\mathfrak{B}_2\}}M^T_{\{\mathfrak{B}_1\}}$. Fortunately the matrix is already almost in reduced row echelon form. By moving the row of zeros to the bottom we obtain the row reduced matrix,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So, to find all vectors $\vec{x}$ so that $\left[_{\{\mathfrak{B}_2\}}M^T_{\{\mathfrak{B}_1\}}\right]\vec{x} = \vec{0}$ involves reading off solutions to the augmented matrix,

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Since there are only three columns we know that any vector in the kernel must be three dimensional. Thus we can immediately read off the solution $\vec{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, and so the kernel is only the zero vector. Which makes sense. What polynomial gets killed off by multiplying it by x? Only the zero polynomial. Thus we would expect $Ker(_{\{\mathfrak{B}_2\}}M^T_{\{\mathfrak{B}_1\}}) = \{\vec{0}\}$.

To find $Im(_{\{\mathfrak{B}_2\}}M^T_{\{\mathfrak{B}_1\}})$ we only need to look at the columns with pivotal 1's. Since this is every column we have $Im(_{\{\mathfrak{B}_2\}}M^T_{\{\mathfrak{B}_1\}}) = \textbf{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}$. Notice that I took the columns from the *original non row reduced* matrix. This result also makes intuitive sense. What kind of polynomial would you expect to obtain from multiplying a polynomial by x? You will get a polynomial one degree higher with no constant term. This is exactly what $Im(_{\{\mathfrak{B}_2\}}M^T_{\{\mathfrak{B}_1\}})$ tells us.

Problem 3 Let $\mathfrak{B}_1 = \{1, x, x^2\}$ be a basis for $\mathcal{P}_2$, and let $\mathfrak{B}_2 = \{1, x, x^2, x^3\}$ be a basis for $\mathcal{P}_3$. Let $\heartsuit : \mathcal{P}_2 \to \mathcal{P}_3$ be the map defined by,

$$\heartsuit(p(x)) = \int_0^x p(t)dt$$

i) Find ${}_{\{\mathfrak{B}_2\}}M^{\heartsuit}_{\{\mathfrak{B}_1\}}$
ii) Use ${}_{\{\mathfrak{B}_2\}}M^{\heartsuit}_{\{\mathfrak{B}_1\}}$ to compute $\heartsuit(3x^2 + 2x + 1)$. Note, all in all you will need to compute

$$\Phi_{\{\mathfrak{B}_2\}}([{}_{\{\mathfrak{B}_2\}}M^{\heartsuit}_{\{\mathfrak{B}_1\}}]\Psi_{\{\mathfrak{B}_1\}}(p(x)))$$

iii) Find $Ker({}_{\{\mathfrak{B}_2\}}M^{\heartsuit}_{\{\mathfrak{B}_1\}})$ and $Im({}_{\{\mathfrak{B}_2\}}M^{\heartsuit}_{\{\mathfrak{B}_1\}})$.

Solution : Granted this problem looks really gross, but it's not much different from the previous problem. The point of this exercise was to try to get you to realize that the computation involved in this material is not at all bad. The hard part is translating the %$&? notation!
ii) We are looking to find,

$${}_{\{\mathfrak{B}_2\}}M^{\heartsuit}_{\{\mathfrak{B}_1\}} = \begin{bmatrix} | & | & | \\ \Psi_{\mathfrak{B}_2}(\heartsuit(1)) & \Psi_{\mathfrak{B}_2}(\heartsuit(x)) & \Psi\mathfrak{B}_2(\heartsuit(x^2)) \\ | & | & | \end{bmatrix}$$

So,

$$\Psi_{\mathfrak{B}_2}(\heartsuit(1)) = \Psi_{\mathfrak{B}_2}(\int_0^x 1dt) = \Psi_{\mathfrak{B}_2}(x) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Psi_{\mathfrak{B}_2}(\heartsuit(x)) = \Psi_{\mathfrak{B}_2}(\int_0^x tdt) = \Psi_{\mathfrak{B}_2}(\frac{1}{2}x^2) = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

$$\Psi_{\mathfrak{B}_2}(\heartsuit(x^2)) = \Psi_{\mathfrak{B}_2}(\int_0^x t^2dt) = \Psi_{\mathfrak{B}_2}(\frac{1}{3}x^3) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{3} \end{pmatrix}$$

Thus, we have,

$${}_{\{\mathfrak{B}_2\}}M^{\heartsuit}_{\{\mathfrak{B}_1\}} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

ii) We are looking to find $\heartsuit(3x^2+2x+1)$. The directions say that we need to use the matrices, but let's use the definition of $\heartsuit$ to see what we had better obtain.

$$\heartsuit(3x^2+2x+1) = \int_0^x 3t^2+2t+1dt$$
$$= \left[\frac{3}{3}t^3 + \frac{2}{2}t^2 + t\right]_0^x$$
$$x^3+x^2+x$$

Boy, that sure does look nice. It almost seems like that had been planned to make the students lives a little easier. But I digress. According to the rules (directions) we are to find,

$$\Phi_{\{\mathfrak{B}_2\}}([_{\{\mathfrak{B}_2\}}M^{\heartsuit}_{\{\mathfrak{B}_1\}}]\Psi_{\{\mathfrak{B}_1}(p(x))$$

So, lets start at the beginning.

$$\Psi_{\{\mathfrak{B}_1}(3x^2+2x+1) = \begin{pmatrix}1\\2\\3\end{pmatrix}$$

So,

$$\left[_{\{\mathfrak{B}_2\}}M^{\heartsuit}_{\{\mathfrak{B}_1\}}\right]\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{bmatrix}0&0&0\\1&0&0\\0&\frac{1}{2}&0\\0&0&\frac{1}{3}\end{bmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}0\\1\\1\\1\end{pmatrix}$$

So, we finally have,

$$\Phi_{\mathfrak{B}_2}\left(\begin{pmatrix}0\\1\\1\\1\end{pmatrix}\right) = 0(1)+1(x)+1(x^2)+1(x^3) = x+x^2+x^3$$

How 'bout that! It worked!

iii) Finally, we need to find $Ker\left(_{\{\mathfrak{B}_2\}}M^{\heartsuit}_{\{\mathfrak{B}_1\}}\right)$ and $Im\left(_{\{\mathfrak{B}_2\}}M^{\heartsuit}_{\{\mathfrak{B}_1\}}\right)$. This is almost identical to the second problem. We see that

$$Ker\left(_{\{\mathfrak{B}_2\}}M^{\heartsuit}_{\{\mathfrak{B}_1\}}\right) = \{\vec{0}\}$$

and

$$Im\left(_{\{\mathfrak{B}_2\}}M^{\heartsuit}_{\{\mathfrak{B}_1\}}\right) = \textbf{span}\left\{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\\frac{1}{2}\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\\frac{1}{3}\end{pmatrix}\right\}$$