Math 2601 C2
Homework 5
Version 2.0

Please do all three of the following problems and email me if you need any assistance (mullikin@math.gatech.edu). The problems are to be turned in **Monday Feb 12, 2001** at 2:05pm. Please, if your work is more than one page, find some mechanical means of maintaining some type of connectedness between the pages. Also, please write neatly. If I can't read your work, I can't give you any credit.

Problem 1 Before working this problem, please see the supplement on the web about finding inverse matrices. Consider the following 4×4 matrix.

$$A = \begin{bmatrix} 4 & -2 & 2 & -1 \\ 0 & -1 & 0 & 2 \\ 4 & -3 & 2 & 1 \\ 8 & -5 & 3 & 0 \end{bmatrix}$$

i) Either perform row operations on A or compute $detA$ to verify that A is invertible.
ii) Compute A^{-1}, and compute AA^{-1} to verify your solution is correct.

Solution : I will expand across the second row, because it has two zeros in it.

i)

$$det(A) = 0(-1)^{2+1}det\begin{bmatrix} -2 & 2 & 1 \\ -3 & 2 & 1 \\ -5 & 3 & 0 \end{bmatrix} + -1(-1)^{2+2}det\begin{bmatrix} 4 & 2 & 1 \\ 4 & 2 & 1 \\ 8 & 3 & 0 \end{bmatrix}$$

$$+0(-1)^{2+3}det\begin{bmatrix} 4 & -2 & 1 \\ 4 & -3 & 1 \\ 8 & -5 & 0 \end{bmatrix} + 2(-1)^{2+4}det\begin{bmatrix} 4 & -2 & 2 \\ 4 & -3 & 2 \\ 8 & -5 & 3 \end{bmatrix}$$

$$= -det\begin{bmatrix} 4 & 2 & 1 \\ 4 & 2 & 1 \\ 8 & 3 & 0 \end{bmatrix} + 2det\begin{bmatrix} 4 & -2 & 2 \\ 4 & -3 & 2 \\ 8 & -5 & 3 \end{bmatrix}$$

$$= -1\left[1(-1)^{1+3}det\begin{bmatrix} 4 & 2 \\ 8 & 3 \end{bmatrix} + 1(-1)^{2+3}det\begin{bmatrix} 4 & 2 \\ 8 & 3 \end{bmatrix} + 0(-1)^{3+3}det\begin{bmatrix} 4 & 2 \\ 4 & 2 \end{bmatrix}\right]$$

$$+2\left[2(-1)^{3+1}det\begin{bmatrix} 4 & -3 \\ 8 & 5 \end{bmatrix} + 2(-1)^{2+3}det\begin{bmatrix} 4 & -2 \\ 8 & 5 \end{bmatrix} + 3(-1)^{3+3}det\begin{bmatrix} 4 & -2 \\ 4 & -3 \end{bmatrix}\right]$$

$$= -1\left[(1(12-16) - 1(12-16)\right]$$

$$+2\left[2(20+24) - 2(20+16) + 3(-12+8)\right]$$

$$= 0 + 2(88 - 72 - 12)$$

$$= 8$$

Since $det(A) \neq 0$ we know that A has full rank, and is invertible.

ii) You should check, using row reduction, that

$$A^{-1} = \begin{bmatrix} \frac{5}{8} & \frac{3}{8} & -\frac{11}{8} & \frac{1}{2} \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$$

Problem 2 Let $\mathfrak{B}_1 = \{1, x, x^2\}$ be the usual basis in $\mathcal{P}_2$.
i) Check that the set $\mathfrak{B}_2 = \{x^2 - 2x, x^2 + 1, x - 1\}$ is also a basis of $\mathcal{P}_2$.
ii) Compute the change of basis matrix from $\mathfrak{B}_1$ to $\mathfrak{B}_2$ and from $\mathfrak{B}_2$ to $\mathfrak{B}_1$. Check that these two matrices are inverses of each other.

Solution :

i) It suffices to show that the matrix formed by columns of the coordinates (in the usual basis) of the vectors in $\mathfrak{B}_2$ has nonzero determinent. Since this matrix will necessarily have full rank, thus each column will be pivotal, thus the columns are independent. Since we would then have three independent vectors in a three dimensional space, it would then follow that $\mathfrak{B}_2$ is a basis. So,

$$\Psi_{\mathfrak{B}_1}(x^2 - 2x) = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}, \Psi_{\mathfrak{B}_1}(x^2 + 1) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \Psi_{\mathfrak{B}_1}(x - 1) = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

And we obtain the matrix,

$$V = \begin{bmatrix} 0 & 1 & -1 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

After a computation we find that $det(V) = 3 \neq 0$. So we deduce that V has full rank, so it must have linearly independent columns. Since the columns corrsespond to $\mathfrak{B}_2$ in the usual basis, we know that $\mathfrak{B}_2$ is a basis for $\mathcal{P}_2$.

ii) We are looking for ${}_{\mathfrak{B}_2}M^I_{\mathfrak{B}_1}$ and ${}_{\mathfrak{B}_1}M^I_{\mathfrak{B}_2}$. Let's do the difficult one first.

$${}_{\mathfrak{B}_2}M^I_{\mathfrak{B}_1} = \begin{bmatrix} | & | & | \\ \Psi_{\mathfrak{B}_2}(1) & \Psi_{\mathfrak{B}_2}(x) & \Psi_{\mathfrak{B}_2}(x^2) \\ | & | & | \end{bmatrix}$$

Continuing, we have

$$\Psi_{\mathfrak{B}_2}(1) = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}$$

where

$$1 = a_1(x^2 - 2x) + b_1(x^2 + 1) + c_1(x - 1)$$

This implies that we need to solve the system of equations,

$$\begin{aligned} 0a_1 + 1b_1 - 1c_1 &= 1 \\ -2a_1 + 0b_1 + 1c_1 &= 0 \\ 1a_1 + 1b_1 + 0c_1 &= 0 \end{aligned}$$

This can be done using row reduction, so you should check and verify that $\Psi_{\mathfrak{B}_2}(1) = \begin{pmatrix} -\frac{1}{3} \\ \\ \frac{1}{3} \\ \\ -\frac{2}{3} \end{pmatrix}$

Through similar reasoning we have $\Psi_{\mathfrak{B}_2}(x) = \begin{pmatrix} -\frac{1}{3} \\ \\ \frac{1}{3} \\ \\ \frac{1}{3} \end{pmatrix}$ and $\Psi_{\mathfrak{B}_2}(x^2) = \begin{pmatrix} \frac{1}{3} \\ \\ \frac{2}{3} \\ \\ \frac{2}{3} \end{pmatrix}$.

Thus, we have

$$_{\mathfrak{B}_2}M^I_{\mathfrak{B}_1} = \begin{bmatrix} -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\ \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Finding $_{\mathfrak{B}_2}M^I_{\mathfrak{B}_1}$ is significantly easier, since we are translating from a bizarre basis into the standard one. We proceed in the same way,

$$_{\mathfrak{B}_1}M^I_{\mathfrak{B}_2} = \begin{bmatrix} | & | & | \\ \Psi_{\mathfrak{B}_1}(x^2 - 2x) & \Psi_{\mathfrak{B}_1}(x^2 + 1) & \Psi_{\mathfrak{B}_1}(x - 1) \\ | & | & | \end{bmatrix}$$

But here, computing the columns is significantly easier. For example, $\Psi_{\mathfrak{B}_1}(x^2 - 2x) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ where, $x^2 - 2x = a(1) + b(x) + c(x^2)$. So we can immediately read off, $a = 0, b = -2, c = 1$ and we have found the first columns! So, you should verify, we obtain,

$$_{\mathfrak{B}_1}M^I_{\mathfrak{B}_2} = \begin{bmatrix} 0 & 1 & -1 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

To see that they are inverses of one another, just check and see that $\left[{}_{\mathfrak{B}_2}M^I_{\mathfrak{B}_1}\right]\left[{}_{\mathfrak{B}_1}M^I_{\mathfrak{B}_2}\right] = I$, by performing the matrix multiplication.

Problem 3 Let

$$\mathfrak{B}_1 = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\}$$

and

$$\mathfrak{B}_2 = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

i) Verify that the sets are bases in $\mathbb{R}^3$.
ii) Write the vector $\vec{u} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ in the $\mathfrak{B}_1$ basis.
iii) Find the change of basis matrix from $\mathfrak{B}_1$ to $\mathfrak{B}_2$.
iv) Find the change of basis matrix from $\mathfrak{B}_2$ to $\mathfrak{B}_1$.
v) Let $\vec{a} = 2\vec{v}_1 - 2\vec{v}_2 + 3\vec{v}_3$. Write this vector in the $\mathfrak{B}_2$ basis (i.e. find $\Psi_{\mathfrak{B}_2}(\vec{a})$). Check the result by writing $\vec{a}$ in the standard basis and check that both representations really give the same vector.

i) Let

$$V = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

and

$$W = \begin{bmatrix} -1 & 2 & 1 \\ 1 & -1 & -1 \\ 0 & 2 & 1 \end{bmatrix}$$

Then notice that $det(V) = -1$ and $det(W) = -1$. Since neither matrix had determinent zero we know that each matrix has full rank. Thus all the columns are independent, thus the columns form a basis (since there are three of them and we are dealing with $\mathbb{R}^3$).

ii) We will proceed as in example problem 5.12 on page 62 of [Notes:TH]. What we are looking for is a representation for $\vec{u} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ as a linear combination of the elements in $\mathfrak{B}_1$. That is, we want to find a, b, and c so that,

$$\vec{u} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = a \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + c \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Or, to use terminology we have seen before, we are looking for $\Psi_{\mathfrak{B}_1}(\vec{(u})$. As we have seen in class this can be found using the inverse of the matrix V. That is,

$$\Psi_{\mathfrak{B}_1}(\vec{u}) = V^{-1}\vec{u} = \begin{bmatrix} 5 & -2 & 1 \\ -7 & 3 & -1 \\ -4 & 2 & -1 \end{bmatrix} \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} (15-8+1) \\ (-21+12-1) \\ (-12+8-1) \end{pmatrix}$$
$$= \begin{pmatrix} 8 \\ -10 \\ -5 \end{pmatrix}$$

I left out the part where we found the inverse. It's all done the same way, you should verify that it is correct. Now, let's see if we got what we should have.

$$8\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} - 10\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - 5\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \vec{u}$$

How 'bout that, it worked!

iii) As we saw in class, and as written in Theorem 5.13 of [Notes:TH] we see that,

$${}_{\mathfrak{B}_2}M^I_{\mathfrak{B}_1} = W^{-1}V$$

where, V is as in part ii) and

$$W = \begin{bmatrix} -1 & 2 & 1 \\ 1 & -1 & -1 \\ 0 & 2 & 1 \end{bmatrix}$$

After some computation, we find that

$$W^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix}$$

Thus,

$${}_{\mathfrak{B}_2}M^I_{\mathfrak{B}_1} = W^{-1}V = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 2 & -2 \\ 4 & 1 & 3 \\ -6 & 0 & -7 \end{bmatrix}$$

iv) Similar to part iii) we see that we are looking for,

$${}_{\mathfrak{B}_1}M^I_{\mathfrak{B}_2} = V^{-1}W = \begin{bmatrix} 5 & -2 & 1 \\ -7 & 3 & -1 \\ -4 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 1 \\ 1 & -1 & -1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -7 & 14 & 8 \\ 10 & -19 & -11 \\ 6 & -12 & -7 \end{bmatrix}$$

v) We have $\vec{a} = 2\vec{v}_1 - 2\vec{v}_2 + 3\vec{v}_3$. So, $\Psi_{\mathfrak{B}_1}(\vec{a}) = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$. So, to find $\Psi_{\mathfrak{B}_2}(\vec{a})$ we only need to perform the multiplication,

$$\Psi_{\mathfrak{B}_2}(\vec{a}) = W^{-1}V\Psi_{\mathfrak{B}_1}(\vec{a}) = \begin{bmatrix} 1 & 2 & -2 \\ 4 & 1 & 3 \\ -6 & 0 & -7 \end{bmatrix} \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -8 \\ 15 \\ -33 \end{pmatrix}$$

Indeed,

$$-8\vec{w}_1 + 15\vec{w}_2 - 33\vec{w}_3 = -8 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + 15 \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} - 33 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ -3 \end{pmatrix}$$

As if by magic, notice also,

$$\vec{a} = 2\vec{v}_1 - 2\vec{v}_2 + 3\vec{v}_3 = 2 \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ -3 \end{pmatrix}$$