Math 2601 C2
Homework 7

Please do all five of the following problems and email me if you need any assistance (mullikin@math.gatech.edu). The problems are to be turned in Friday March 2, 2001 at 2:05pm. You may turn it in early (notice early $\neq$ late) if for some reason you will not be in class on Friday, although I can't imagine anyone voluntarily missing one of my stellar lectures. Please please please, staple your work if it is more than one page (please). Also, you must write neatly. If I can't read your work, I can't give you any credit.

Let $x_i(t) : \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable functions for $i = 1, 2, 3$. Consider the following system of differential equations.

$$\mathbb{S} = \left\{ \begin{array}{c} x_1'(t) = 5x_1(t) + 2x_2(t) + x_3(t) \\ x_2'(t) = x_1(t) + 4x_2(t) - x_3(t) \\ x_3'(t) = -x_1(t) - 2x_2(t) + 3x_3(t) \end{array} \right\}$$

Problem 1 Find a matrix A so that $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$, where $\vec{\mathbf{x}}'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix}$ and $\vec{\mathbf{x}}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$.

Solution :

$$A = \begin{bmatrix} 5 & 2 & 1 \\ 1 & 4 & -1 \\ -1 & -2 & 3 \end{bmatrix}$$

Problem 2 Find the roots of the characteristic polynomial [1] for A, thus obtaining the eigenvalues.

Solution :

First we need to compute the characteristic polynomial.

$$\begin{aligned}\det(A-\lambda I) &= (5-\lambda)\left[(4-\lambda)(3-\lambda)-2\right]-1\left[2(3-\lambda)+2\right]+-1\left[-2-(4-\lambda)\right]\\ &= (5-\lambda)\left[10-7\lambda+\lambda^2\right]-\left[8-2\lambda\right]+\left[6-\lambda\right]\\ &= 50-35\lambda+5\lambda^2-10\lambda+7\lambda^2-\lambda^3+\lambda-2\\ &= -\lambda^3+12\lambda^2-44\lambda+48\end{aligned}$$

Next, we need to find the roots of this polynomial. This is where the hint will come in. We know that if there are any rational roots they will be one of the following, $\pm\frac{48}{1}, \pm\frac{24}{1}, \pm\frac{16}{1}, \pm\frac{12}{1}, \pm\frac{8}{1}, \pm\frac{6}{1}, \pm\frac{4}{1}, \pm\frac{3}{1}, \pm\frac{2}{1}, \text{or} \pm\frac{1}{1}$. After some trial and error, we can see that the roots are $\lambda_1 = 2$, $\lambda_2 = 4$, and $\lambda_3 = 6$. Thus, we have found the eigenvalues for A.

[1] Given a polynomial equation $0 = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, if there are any rational roots, then they will be of the form $\pm\frac{x_i}{y_j}$, where $\{x_i\}$ is the set of all divisors of $|a_0|$, and $\{y_j\}$ is the set of all divisors of $|a_n|$.

Problem 3 Find an explicit solution for the system $\mathbb{S}$, with the initial conditions $x_1(0) = 1$, $x_2(0) = 2$, and $x_3(0) = -1$.

Solution :

We know, from the notes, that a solution to the system $\mathbb{S}$ will be $\vec{x}(t) = e^{At}\vec{x}(0)$. So, we only need to compute e^{At}. By definition we know that $e^{At} = Ve^{Dt}V^{-1}$, where VDV^{-1} is a diagonlaization of A. So, it seems we will need to find the eigenvectors for A as well.

Case 1 : $\lambda_1 = 2$.

$$\begin{bmatrix} 3 & 2 & 1 & \mid & 0 \\ 1 & 2 & -1 & \mid & 0 \\ -1 & -2 & 1 & \mid & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & -1 & \mid & 0 \\ 1 & 0 & 1 & \mid & 0 \\ 0 & 0 & 0 & \mid & 0 \end{bmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

Case 2 : $lambda_2 = 4$.

$$\begin{bmatrix} 1 & 2 & 1 & \mid & 0 \\ 1 & 0 & -1 & \mid & 0 \\ -1 & -2 & -1 & \mid & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 1 & \mid & 0 \\ 1 & 0 & -1 & \mid & 0 \\ 0 & 0 & 0 & \mid & 0 \end{bmatrix} \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Case 3 : $lambda_3 = 6$.

$$\begin{bmatrix} -1 & 2 & 1 & \mid & 0 \\ 1 & -2 & -1 & \mid & 0 \\ -1 & -2 & -3 & \mid & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 & \mid & 0 \\ 1 & 0 & 1 & \mid & 0 \\ 0 & 1 & 1 & \mid & 0 \end{bmatrix} \Rightarrow \vec{v}_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Thus, we have

$$A = \begin{bmatrix} 5 & 2 & 1 \\ 1 & 4 & -1 \\ -1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}^{-1}$$

So, our solution will be of the form,

$$\begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{4t} & 0 \\ 0 & 0 & e^{6t} \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$= \begin{bmatrix} \frac{3}{2}e^{6t} - \frac{1}{2}e^{2t} \\ \\ \frac{3}{2}e^{6t} + \frac{1}{2}e^{2t} \\ \\ -\frac{3}{2}e^{6t} + \frac{1}{2}e^{2t} \end{bmatrix}$$

Problem 4 Compute A^{170574}. Hint: You are *really* close to having a diagonalization for A.

Solution :

$$A^{170574} = \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2^{170574} & 0 & 0 \\ 0 & 4^{170574} & 0 \\ 0 & 0 & 6^{170574} \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}^{-1}$$

Problem 5 Would you be interested in a homework assignment over spring break? The possible answers to this question are either "Boy howdy! You bet I would!" or "No thanks Mr. Sadist. I would rather have a spring *break*, not a spring *work*.".

Anti - Solution :

This is a problem asking for a preference. There is no right or wrong. (Oooo.... how I love waxing philosophic!)