An example

Lets consider a linear operator $T : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ where $T(p(x)) = \int_0^x p(t)dt$. Let the basis for $\mathcal{P}_1$ and $\mathcal{P}_2$ be $\mathfrak{B}_1 = \{1+x, 1-x\}$ and $\mathfrak{B}_2 = \{1+x, 1+x^2, x+x^2\}$ respectively. The goal is to find the matrix ${}_{\{\mathfrak{B}_2\}}M^T_{\{\mathfrak{B}_1\}}$.

Recall that the matrix has the form,

$$
{}_{\{\mathfrak{B}_2\}}M^T_{\{\mathfrak{B}_1\}} = \begin{bmatrix} | & | \\ \Psi_{\{\mathfrak{B}_2\}}(T(1+x)) & \Psi_{\{\mathfrak{B}_2\}}(T(1-x)) \\ | & | \end{bmatrix}
$$

So, first we need to find $T(x+1)$ and $T(1-x)$.

$$
\begin{aligned}
T(x+1) &= \int_0^x (t+1)dt = \frac{1}{2}x^2 + x \\
T(1-x) &= \int_0^x (1-t)dt = x - \frac{1}{2}x^2
\end{aligned}
$$

Next we need to find out what these vectors look like in terms of coordinates. That is, find $\Psi_{\{\mathfrak{B}_2\}}(\frac{1}{2}x^2 + x)$. Well, we know, by definition,$\Psi_{\{\mathfrak{B}_2\}}(\frac{1}{2}x^2 + x) = (c_1, c_2, c_3)$ where c_1, c_2, and c_2 satisfy the equation,

$\frac{1}{2}x^2 + x = c_1(1+x) + c_2(1+x^2) + c_3(x+x^2)$

Multiplying this out we see that we have,

$$
\tfrac{1}{2}x^2 + x = (c_1 + c_2) + (c_1 + c_3)x + (c_2 + c_3)x^2
$$

So we must solve the system of equations,

$$
\begin{aligned}
c_1 + c_2 &= 0 \\
c_1 + c_3 &= 1 \\
c_2 + c_3 &= \tfrac{1}{2}
\end{aligned}
$$

so that all of the coefficients of 1, x, and x^2 match up. Bring on the row reduction!

$$
\begin{aligned}
&\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & \frac{1}{2} \end{bmatrix}
&& (R_2 = R_2 + (-1)R_1) \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & 1 & \frac{1}{2} \end{bmatrix} \\
(R_3 = R_3 + R_2) &\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 2 & \frac{3}{2} \end{bmatrix}
&& (\tfrac{1}{2}R_3) \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & \frac{3}{4} \end{bmatrix} \\
(R_2 = R_2 + (-1)R_3) &\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & \frac{3}{4} \end{bmatrix}
&& (R_1 = R_1 + R_2) \begin{bmatrix} 1 & 0 & 0 & \frac{1}{4} \\ 0 & -1 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & \frac{3}{4} \end{bmatrix}
\end{aligned}
$$

So we have $c_1 = \frac{1}{4}$, $c_2 = -\frac{1}{4}$, and $c_3 = \frac{3}{4}$. You can check and see that $\frac{1}{4}(1+x) - \frac{1}{4}(1+x^2) + \frac{3}{4}(x+x^2) = x + \frac{1}{2}x^2$ as desired. Thus we have found the first column of the matrix ${}_{\{\mathfrak{B}_2\}}M^T_{\{\mathfrak{B}_1\}}$ to be $\begin{pmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ \frac{3}{4} \end{pmatrix}$. We find the second column in the same way. We know $T(1-x) = x - \frac{1}{2}x^2$ and so, after solving another system of equations, we have $\Psi_{\{B_2\}} = (\frac{3}{4}, -\frac{3}{4}, \frac{1}{4})$. So, finally,

$$
{}_{\{\mathfrak{B}_2\}}M^T_{\{\mathfrak{B}_1\}} = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ -\frac{1}{4} & -\frac{3}{4} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix}
$$

Let us test our result on an arbitrary polynomial $a + bx \in \mathcal{P}_2$. Remember, first we need to compute to coordinates of $a+bx$ in the basis $\mathfrak{B}_1$. So, we readily see that $a + bx = \frac{a+b}{2}(1+x) + \frac{a-b}{2}(1-x)$ by solving a system of two equations as we did above to find the coordinates. So, to find $T(a+bx)$ we perform the matrix multiplication,

$$
\begin{aligned}
&{}_{\{\mathfrak{B}_2\}}M^T_{\{\mathfrak{B}_1\}} \begin{bmatrix} \frac{a+b}{2} \\ \frac{a-b}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} \frac{a+b}{2} \\ \frac{a-b}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{2a-b}{4} \\ \frac{-2a+b}{4} \\ \frac{2a+b}{4} \end{bmatrix}
\end{aligned}
$$

Looks icky right? Remember the output of this matrix multiplication is in the bizarro basis $\mathfrak{B}_2$. To translate into something more friendly we need to see what this looks like in the standard basis. So, we have the coordinates and the basis. Multiply it out.

$$
\begin{aligned}
&\frac{2a-b}{4}(1+x) \quad + \quad \frac{-2a+b}{4}(1+x^2) \quad + \quad \frac{2a+b}{4}(x+x^2) \\
&= \frac{2a-b+-2a+b}{4} \quad + \quad \frac{2a-b+2a+b}{4}x \quad + \quad \frac{-2a+b+2a+2}{4}x^2 \\
&= ax + \frac{b}{2}x^2
\end{aligned}
$$

Do you believe it? Sure, the transformation took a polynomial to its integral evaluated at x. So, $T(a+bx) = \int_0^x (a+bt)dt = ax + \frac{b}{2}x^2$. How 'bout 'dem apples!