**Finding Matrix Inverses:**
**Elementary Matrices**

The questions at hand are, if I am given a general $n \times n$ matrix $A$, when can I compute its inverse, and how do I do it? Before we answer these questions, we must answer a question that is hiding in the background. Specifically, what do we mean when we say inverse.

**Definition :** Let $A$ be an $n \times n$ matrix, if there exists another $n \times n$ matrix $B$ so that $BA = AB = \mathbf{I}$ where $\mathbf{I}$ denotes the $n \times n$ identity matrix, then we say $B$ is the inverse of $A$ and we write $B = A^{-1}$.

Notice, that in the above definition I am insisting that both $A$ and $A^{-1}$ be square matrices. Then, with this restriction, it is true that if $A^{-1}$ exists, then it is unique. So, when can we find an inverse of a matrix? Recall the pictures drawn on the board during class where I was describing the kernel and image of a given linear transformation. I mumbled something about an inverse existing if and only if the kernel is empty. Why would this be? Well, suppose it wasn't empty. That is suppose that $A\vec{x} = \vec{0}$ for some $\vec{x} \neq \vec{0}$. Notice also, since $A$ is a linear transformation, we know $A\vec{0} = \vec{0}$. Now, try to describe where to send $A^{-1}(\vec{0})$. Does it go to $\vec{x}$ or does it go to $\vec{0}$? How do you choose? The problem is we cannot come up with a consistent way to send one point to many (this is like a function not passing the vertical line test). So, in order for an inverse to even make an inkling of sense, we must require that the only element in the kernel of an invertible matrix is zero. Is that enough? Well, that along with the condition that the matrix be square, yes. So, if a square matrix has empty kernel, then it must be able to be row reduced to the identity matrix right. Thus, it will have a pivotal 1 in each column and so the matrix will have full rank. That is, is $A$ is an invertible $n \times n$ matrix, then $ImA = \mathbb{R}^n$. A nice way to determine if a square matrix is invertible or not is to take a determinant. Since, an $n \times n$ matrix will have nonzero determinant if and only if its kernel only contains the zero vector. I.e. $det(A) \neq 0$ if and only if $A$ has full rank if and only if $A$ is invertible. Notice that if $A$ is a change of basis matrix from $\mathbb{R}^n$ into $\mathbb{R}^n$ then $A$ must have full rank (since it sends one basis onto another basis) and is therefore invertible.

Now we need to answer the second question. How do we find the inverse. You may recall, from 1502, the notion of elementary matrices. An elementary matrix is a matrix that represents one elementary row operation on the identity matrix. For example, the elementary matrix representing the row operation $(R_2 = R_2 + (-3)R_1)$ on a $3 \times 3$ matrix is the matrix,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix representing swapping row three with row two would be,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Get the idea? So what? "Why is this nice?", you may inquire. Suppose I wanted to keep track of all the row operations I perform on a given matrix. Then, I would only need to keep a list of the elementary matrices. Better than that, I can keep all the information involved in the row reduction by multiplying the elementary matrices together. *They must be multiplied in a proper order from left to right with the left most matrix being the row operation last performed and the right most matrix being the first row reduction operation performed!* Let's look at an example. Consider the matrix,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 1 & 0 \end{bmatrix}$$

Now lets perform row operations and reduce $A$ to the identity.

$$\begin{array}{rcrc} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 1 & 0 \end{bmatrix} & (R_2 = R_2 + (-3)R_3) & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 1 & 0 \end{bmatrix} \\ (R_2 \leftrightarrow R_3) & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} & (\frac{1}{4}R_3) & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Now, what elementary matrices do we have? We already know two. The third is the matrix,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

So, if I wanted to write down all of the row reduction steps I used I would find the matrix product,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{1}{4} & -\frac{3}{4} \end{bmatrix}$$

Wow... Neat... So what? So, I claim we have found,

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{1}{4} & -\frac{3}{4} \end{bmatrix}$$

Let's check it out.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{1}{4} & -\frac{3}{4} \end{bmatrix} = \begin{bmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ 0 & 0+0+4\frac{1}{4} & 0+3+4\frac{-3}{4} \\ 0+0+0 & 0+0+0 & 0+1+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

How cool is that?! So to find inverses we only need to keep track of the elementary operations involved in reducing to the identity. It turns out there is a really easy way to do that. Given a matrix $A$ put the identity matrix next to it. Then whenever you do a row operation on $A$ do the same row operation on the identity. This way you will keep track of the product of elementary matrices without even having to multiply! When $A$ has been reduced to the identity then the other matrix will have been changed into $A^{-1}$. Let's do an example with the same matrix above to illustrate.

$$\begin{bmatrix} 1 & 0 & 0 & | & 1 & 0 & 0 \\ 0 & 3 & 4 & | & 0 & 1 & 0 \\ 0 & 1 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$$(R_2 = R_2 + (-3)R_3) \begin{bmatrix} 1 & 0 & 0 & | & 1 & 0 & 0 \\ 0 & 0 & 4 & | & 0 & 1 & -3 \\ 0 & 1 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$$(R_2 \leftrightarrow R_3) \begin{bmatrix} 1 & 0 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & 0 & 0 & 1 \\ 0 & 0 & 4 & | & 0 & 1 & -3 \end{bmatrix}$$

$$(\frac{1}{4}R_3) \begin{bmatrix} 1 & 0 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & 0 & 0 & 1 \\ 0 & 0 & 1 & | & 0 & \frac{1}{4} & -\frac{3}{4} \end{bmatrix}$$

Neat huh? This result can be generalized to $n \times n$ matrices for any $n$. So, to sum up, to find the inverse of a matrix first see if it exists. To do this just take a determinant. If the determinant is ***not*** zero then the inverse exists. Stick the identity matrix next to the one you are trying to invert. Perform row operations on the matrix you are trying to invert until it is brought to the identity matrix, all the while performing the same operations on the appended identity matrix. When you are done row reducing, the appended identity matrix will have been transformed into the inverse matrix. Magic!