Determinants
Fear Not!

Let's start with a definition. Be warned the definition is nasty, and is *very* hard to get a good intuition as to what is going on. After we have defined everything we'll go back and see if we can untangle the mess and work some examples.

Definition : First, define the determinant of the 1×1 matrix $A = [a_{11}]$ to be $det(A) = a_{11}$. We now define the determinant of an $n \times n$ matrix A inductively. So, let A be an $n \times n$ matrix and assume the determinant is defined over the set of all $(n-1) \times (n-1)$ matrices. Let A_{ij} denote the $(n-1) \times (n-1)$ matrix obtained by deleting the i^{th} row and j^{th} column from A. Then,

$$det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} det(A_{ij}) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} det(A_{ij})$$

Oh dear, that looks awful. What the equality statements above say is that you can expand a determinant by any row (the sum is indexed by j) or by any column (the sum is indexed by i). Next, what do we mean by A_{ij}? This is best answered with an example. Let A be the following 5×5 matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}$$

Then A_{23} is the 4×4 matrix resulting in deleting the 2^{nd} row and 3^{rd} column.

$$A_{23} = \begin{bmatrix} a_{11} & a_{12} & | & a_{14} & a_{15} \\ - & - & - & - & - \\ a_{31} & a_{32} & | & a_{34} & a_{35} \\ a_{41} & a_{42} & | & a_{44} & a_{45} \\ a_{51} & a_{52} & | & a_{54} & a_{55} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{14} & a_{15} \\ a_{31} & a_{32} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{54} & a_{55} \end{bmatrix}$$

Ok, I think we should do an example. Consider the following 4×4 matrix,

$$B = \begin{bmatrix} 0 & 137 & 632 & 1097 \\ 0 & -5 & 7 & 2 \\ 1 & 1 & -4 & 3 \\ 0 & -11 & 14 & -1 \end{bmatrix}$$

The goal is to compute $det(B)$. So, we will do this two ways. Lets do the blind brute force way first. Then after that we will use our heads a little and compute the determinant in a clever way to arrive at the same answer.

Brute force : expand across the first row and we obtain the following,

$$det(B) = 0(-1)^{1+1}det\left(\begin{bmatrix} -5 & 7 & 2 \\ 1 & -4 & 3 \\ -11 & 14 & -1 \end{bmatrix}\right) + 137(-1)^{1+2}det\left(\begin{bmatrix} 0 & 7 & 2 \\ 1 & -4 & 3 \\ 0 & 14 & -1 \end{bmatrix}\right)$$
$$+632(-1)^{1+3}det\left(\begin{bmatrix} 0 & -5 & 2 \\ 1 & 1 & 3 \\ 0 & -11 & -1 \end{bmatrix}\right) + 1097(-1)^{1+4}det\left(\begin{bmatrix} 0 & -5 & 7 \\ 1 & 1 & -4 \\ 0 & -11 & 14 \end{bmatrix}\right)$$

$$= 0+137(-1)\left[0(-1)^{1+1}det\left(\begin{bmatrix} -4 & 3 \\ 14 & -1 \end{bmatrix}\right) + 7(-1)^{1+2}det\left(\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}\right) + 2(-1)1+3det\left(\begin{bmatrix} 1 & -4 \\ 0 & 14 \end{bmatrix}\right)\right]$$
$$+632(-1)^{1+3}det\left(\begin{bmatrix} 0 & -5 & 2 \\ 1 & 1 & 3 \\ 0 & -11 & -1 \end{bmatrix}\right) + 1097(-1)^{1+4}det\left(\begin{bmatrix} 0 & -5 & 7 \\ 1 & 1 & -4 \\ 0 & -11 & 14 \end{bmatrix}\right)$$

$$= 0 + 137(-1)\left[0(4-42) + 7(-1)(-11-0) + 2(14-0)\right]$$
$$+632(-1)^{1+3}det\left(\begin{bmatrix} 0 & -5 & 2 \\ 1 & 1 & 3 \\ 0 & -11 & -1 \end{bmatrix}\right) + 1097(-1)^{1+4}det\left(\begin{bmatrix} 0 & -5 & 7 \\ 1 & 1 & -4 \\ 0 & -11 & 14 \end{bmatrix}\right)$$

$$= (-137)(35)$$
$$+632\left[0(-1)^{1+1}det\left(\begin{bmatrix} 1 & 3 \\ -11 & -1 \end{bmatrix}\right) + (-5)(-1)^{1+2}det\left(\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}\right) + 2(-1)^{1+3}det\left(\begin{bmatrix} 1 & 1 \\ 0 & -11 \end{bmatrix}\right)\right]$$
$$+1097(-1)^{1+4}det\left(\begin{bmatrix} 0 & -5 & 7 \\ 1 & 1 & -4 \\ 0 & -11 & 14 \end{bmatrix}\right)$$

$$
\begin{aligned}
&= (-137)(35) \\
&+632\left[5(-1-0)+2(-11-0)\right] \\
&+1097(-1)^{1+4} det\left(\begin{bmatrix} 0 & -5 & 7 \\ 1 & 1 & -4 \\ 0 & -11 & 14 \end{bmatrix}\right)
\end{aligned}
$$

$$
= (-137)(35) + (632)(-5-22)
$$

$$
+1097(-1)\left[0(-1)^{1+1} det\left(\begin{bmatrix} 1 & -4 \\ -11 & 14 \end{bmatrix}\right) + (-5)(-1)^{1+2} det\left(\begin{bmatrix} 1 & -4 \\ 0 & 14 \end{bmatrix}\right) + 7(-1)^{1+3} det\left(\begin{bmatrix} 1 & 1 \\ 0 & 11 \end{bmatrix}\right)\right]
$$

$$
= (-137)(35) + (632)(-27) + (1097)(-5(14-0) + 7(11-0))
$$

$$
\begin{aligned}
&= (-137)(35) + (632)(-27) + (1097)(-5(14) + 7(11)) \\
&= (-137)(35) + (632)(-27) + (1097)(7) \\
&= -4795 - 17064 + 7679 \\
&= -14180
\end{aligned}
$$

Smart way : Notice that the definition says that I can calculate the determinant by expanding along any row or column. If I expand along the first column I will only have to compute the determinant of a 3×3 matrix once, since there is only one nonzero term in the first column. That is,

$$
det(B) = 0(-1)^{1+1} det\left(\begin{bmatrix} -5 & 7 & 2 \\ 1 & -4 & 3 \\ -11 & 14 & -1 \end{bmatrix}\right) + 0(-1)^{2+1} det\left(\begin{bmatrix} 137 & 632 & 1097 \\ 1 & -4 & 3 \\ -11 & 14 & -1 \end{bmatrix}\right)
$$

$$
+1(-1)^{3+1} det\left(\begin{bmatrix} 137 & 632 & 1097 \\ -5 & 7 & 2 \\ -11 & 14 & -1 \end{bmatrix}\right) + 0(-1)^{4+1} det\left(\begin{bmatrix} 137 & 632 & 1097 \\ -5 & 7 & 2 \\ 1 & -4 & 3 \end{bmatrix}\right)
$$

$$= 0 + 0 + 1det\left(\begin{bmatrix} 137 & 632 & 1097 \\ -5 & 7 & 2 \\ -11 & 14 & -1 \end{bmatrix}\right) + 0$$

$$= 137(-1)^{1+1}det\left(\begin{bmatrix} 7 & 2 \\ 14 & -1 \end{bmatrix}\right) + 632(-1)^{1+2}det\left(\begin{bmatrix} -5 & 2 \\ -11 & -1 \end{bmatrix}\right) + 1097(-1)^{1+3}det\left(\begin{bmatrix} -5 & 7 \\ -11 & 14 \end{bmatrix}\right)$$

$$= 137(-7 - 28) - 632(5 + 22) + 1097(-70 + 77)$$
$$= 137(-35) - 632(27) + 1097(7)$$
$$= -4795 - 17064 + 7679$$
$$= -14180$$

We got the same answer as before and we did a lot less work! The moral is, when computing determinants, expand along the row or column that has the largest amount of zeros. This way you eliminate the need to do unnecessary computation. Please let me know if this helps or not. If not I will need to do something else. Send your thoughts, suggestion, grievances, etc... to **mullikin@math.gatech.edu** Thanks, and good night.