3 QR factorization revisited

Now we can explain why $A = QR$ factorization is much better when using it to solve $A\mathbf{x} = \mathbf{b}$ than the $A = LU$ factorization obtained even with partial pivoting. Again, we focus on the case of a square matrix, however everything remains true for the general case.

As before, the QR-factorization really splits the problem $A\mathbf{x} = \mathbf{b}$ into two subproblems:

$$Q\mathbf{y} = \mathbf{b} \qquad \text{and} \qquad R\mathbf{x} = \mathbf{y}$$

both of them are easily solvable, since $\mathbf{y} = Q^t\mathbf{b}$ and R is upper triangular.

Since Q is orthogonal, $\|Q\| = 1$ and $\|Q^{-1}\| = \|Q^t\| = 1$, i.e.

$$\text{cond}(Q) = 1 \qquad \text{for orthogonal matrices}$$

Since $\text{cond}(A) \geq 1$ for any matrix, we see that the orthogonal transformations are the most stable ones. In particular, if A is regular, then

$$\|A\| = \|Q\|\,\|R\| \qquad \text{and} \qquad \|A^{-1}\| = \|R^{-1}\|\|Q^t\| \tag{3.1}$$

(PROVE IT(*)!) hence

$$\text{cond}(A) = \text{cond}(QR) = \text{cond}(R)$$

In other words the QR-factorization decouples the original problem $A\mathbf{x} = \mathbf{b}$ into two problems, one of them ($Q\mathbf{y} = \mathbf{b}$) has no error amplification, the other one ($R\mathbf{x} = \mathbf{y}$) has the minimal possible error amplification allowed by the inherent error amplification of the original problem.

The **moral of the story** is that orthogonal transformations are stable, one should use them whenever possible.

3.1 Gram-Schmidt revisited

It is clear that once we have the $A = QR$ decomposition, we do not lose more accuracy than necessary when solving $A\mathbf{x} = \mathbf{b}$. But what about the factorization itself? How good is our algorithm via the Gram-Schmidt procedure?

Again, we assume that A is a square matrix and it is regular. Recall that the Gram-Schmidt procedure is a sequence of multiplications of A from the right by upper triangular matrices (THINK IT OVER (*)), i.e. we have

$$AR_1R_2 \dots R_n = Q,$$

and then, after inverting the R_i matrices, we get

$$A = QR$$

The condition number of the R matrices can be very big, especially if A has some nearly linearly dependent columns. Of course, in this case the condition number of A is big as well. However $\text{cond}(A) = \text{cond}(R) = \text{cond}(R_1R_2 \dots R_n)$ is usually still much smaller than the product of the individual condition numbers $\text{cond}(R_1)\text{cond}(R_2)\dots$. In general it is not "healthy" to perform successive multiplication with ill-conditioned matrices.

The situation is even worse if A is singular or rectangular (which is the general case). Recall that in the original QR-factorization with Gram-Schmidt there was a step when the algorithm had to decide whether the new column is linearly independent from the previous ones or not. The answer to this question is very sensitive to any rounding error.

Recalling the "moral of the story" of the previous section, one should try to use orthogonal transformations. In other words, instead of multiplying A from the right by many unstable upper triangular matrices to bring it into an orthogonal form, we should try to multiply it from the left by a sequence of orthogonal matrices to bring it into an upper triangular form. I.e. we are looking for orthogonal matrices $Q_1, Q_2, \ldots$ such that

$$Q_n Q_{n-1} \ldots Q_2 Q_1 A = R$$

be an upper triangular matrix. Then we can invert all these orthogonal matrices to get $A = QR$. The point is that we invert and successively multiply with stable orthogonal matrices, the rounding errors will not get amplified.

There are two common methods to do this: Householder reflections and Givens rotations. We give a short outline of them.

3.2 Householder reflection

Householder reflection is a matrix of the form $H = I - 2\mathbf{u}\mathbf{u}^t$ where $\|\mathbf{u}\| = 1$. It is the generalization of the reflection onto the plane with normal vector $\mathbf{u}$ in $\mathbf{R}^n$. It is easy to check (CHECK(*)) that $H = H^t$ and $HH^t = H^tH = I$, i.e. it is a symmetric orthogonal matrix.

Given a vector $\mathbf{x} \neq \mathbf{0}$, it is easy to find a Householder reflection $H = I - 2\mathbf{u}\mathbf{u}^t$ to zero out all but the first entry of $\mathbf{x}$, i.e.

$$H\mathbf{x} = \mathbf{x} - 2(\mathbf{u}^t \cdot \mathbf{x})\mathbf{u} = \begin{pmatrix} c \\ 0 \\ 0 \\ \vdots \end{pmatrix} = c\mathbf{e}_1.$$

Since H is orthogonal, $|c| = \|H\mathbf{x}\| = \|\mathbf{x}\|$. Write

$$\mathbf{u} = \frac{1}{2\mathbf{u}^t \cdot \mathbf{x}}(\mathbf{x} - c\mathbf{e}_1);$$

i.e. $\mathbf{u}$ must be parallel to the vector $\tilde{\mathbf{u}} = \mathbf{x} \pm \|\mathbf{x}\|\mathbf{e}_1$ hence

$$\mathbf{u} = \frac{\mathbf{x} \pm \|\mathbf{x}\|\mathbf{e}_1}{\big\|\mathbf{x} \pm \|\mathbf{x}\|\mathbf{e}_1\big\|}$$

One can show that either choice of sign yields a $\mathbf{u}$ satisfying $H\mathbf{x} = c\mathbf{e}_1$ as long as $\widetilde{\mathbf{u}} \neq 0$. We will use $\widetilde{\mathbf{u}} = \mathbf{x} + \text{sign}(x_1)\|\mathbf{x}\|\mathbf{e}_1$ so that there is no cancellation in the first component of $\widetilde{\mathbf{u}}$.

In summary, we get

$$\widetilde{\mathbf{u}} := \begin{pmatrix} x_1 + \text{sign}(x_1)\|\mathbf{x}\| \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \text{with} \qquad \mathbf{u} = \frac{\widetilde{\mathbf{u}}}{\|\widetilde{\mathbf{u}}\|}$$

The corresponding Householder reflection is

$$H(\mathbf{x}) = I - 2\mathbf{u}\mathbf{u}^t = I - \frac{2\widetilde{\mathbf{u}}\widetilde{\mathbf{u}}^t}{\|\widetilde{\mathbf{u}}\|^2}$$

and certainly it is enough to store only the vector $\mathbf{u}$ (or $\widetilde{\mathbf{u}}$) instead of the full Householder matrix $H(\mathbf{x})$. Whenever we have to multiply by $H(\mathbf{x})$, it is always easier to compute the result from the $\mathbf{u}$ vector.

3.3 QR factorization with Householder reflection

The idea is similar to the Gauss elimination in the LU-factorization language. Given an $n \times m$ matrix A (could be rectangular as well), we bring it into an upper triangular form (R) by multiplying it from the left by appropriately chosen Householder matrices. We can assume that none of the columns of A is fully zero (such a column just corresponds to a fully zero column in R, hence it can be removed before we start the algorithm and then can be put back at the end).

In the first step we eliminate all but the top entry in the first column of A. We can do it by one single Householder matrix, namely by $H(\mathbf{a}_1)$ if $\mathbf{a}_1$ is the first column of A. The result is

$$A_1 = H(\mathbf{a}_1)A = \begin{pmatrix} * & * & * & \dots & * \\ 0 & \underline{*} & * & \dots & * \\ 0 & \underline{*} & * & \dots & * \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & \underline{*} & * & \dots & * \end{pmatrix}$$

where $*$ denotes a generic (usually nonzero) entry. Let $H_1 = H(\mathbf{a}_1)$ for brevity.

Next, we look at the second column of the matrix $A_1 = H(\mathbf{a}_1)A$. Cut off the first entry of the second column (since we do not want to change the first row any more), i.e. consider the vector $\widetilde{\mathbf{a}}_2$ of size $(n-1)$ formed from the underlined elements. If this vector is zero, or its only nonzero element is on the top, then already the second column is in upper triangular form, so we can proceed to the next column.

If this cutoff vector $\widetilde{\mathbf{a}}_2$ has nonzero elements below the top entry, then we use a Householder reflection $H(\widetilde{\mathbf{a}}_2)$ in the space of "cutoff" vectors. In the original space this means a multiplication by the matrix

$$H_2 = \left(\begin{array}{c|cccc} 1 & 0 & 0 & \dots & 0 \\ \hline 0 & & & & \\ \vdots & & H(\widetilde{\mathbf{a}}_2) & & \\ 0 & & & & \end{array}\right)$$

The result is

$$A_2 = H_2 A_1 = \begin{pmatrix} * & * & * & * & \dots & * \\ 0 & * & * & * & \dots & * \\ 0 & 0 & \underline{*} & * & \dots & * \\ 0 & 0 & \underline{*} & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \underline{*} & * & \dots & * \end{pmatrix}$$

In the next step we consider the next column of A_2 which has nonzero elements below the third row. Again, cutoff the top two entries of this vector and consider the vector $\widetilde{\mathbf{a}}_3$ of size $(n-2)$ (underlined elements in A_2). We can find a Householder reflection $H(\widetilde{\mathbf{a}}_3)$ in $\mathbf{R}^{n-2}$, and if we multiply A_2 from the left with

$$H_3 = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & H(\tilde{\mathbf{a}}_2) & \\ 0 & 0 & & & \end{array}\right)$$

then the result is

$$A_3 = H_3 A_2 = \begin{pmatrix} * & * & * & * & \dots & * \\ 0 & * & * & * & \dots & * \\ 0 & 0 & * & * & \dots & * \\ 0 & 0 & 0 & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & * & \dots & * \end{pmatrix}$$

After at most $(m-1)$-steps, we clearly arrive at a upper triangular matrix R, hence we have

$$H_{m-1} H_{m-2} \dots H_2 H_1 A = R$$

It is clear that all H_i matrices are orthogonal (they are Householder matrices on a subspace and identity on the complement of that subspace), i.e.

$$A = QR$$

with

$$Q = H_1 H_2 \dots H_{m-2} H_{m-1}$$

(recall that $H_i^{-1} = H_i^t = H_i$)

Notice that we always multiply by orthogonal matrices, which is a stable operation since $\mathrm{cond}(H_i) = 1$.

Problem 3.1 *Find the QR factorization of* $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 2 \\ 2 & 0 & 1 \end{pmatrix}$ *with Householder reflections.*

SOLUTION: The first column vector is $\mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$. The corresponding Householder vector is

$$\widetilde{\mathbf{u}} = \begin{pmatrix} 1+\sqrt{5} \\ 0 \\ 2 \end{pmatrix}$$

with norm square $\|\widetilde{\mathbf{u}}\|^2 = 10 + 2\sqrt{5}$, hence

$$H_1 = H(\mathbf{a}_1) = I - \frac{2\widetilde{\mathbf{u}}\widetilde{\mathbf{u}}^t}{10+2\sqrt{5}} = \begin{pmatrix} -\frac{1}{\sqrt{5}} & 0 & -\frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \\ -\frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \end{pmatrix}$$

and

$$A_1 = H_1 A = \begin{pmatrix} -\sqrt{5} & -\frac{2}{\sqrt{5}} & -\sqrt{5} \\ 0 & 3 & 2 \\ 0 & -\frac{4}{\sqrt{5}} & -\sqrt{5} \end{pmatrix} = \begin{pmatrix} -2.236 & -0.894 & -2.236 \\ 0 & 3 & 2 \\ 0 & -1.788 & -2.236 \end{pmatrix}$$

The next cutoff column vector is

$$\widetilde{\mathbf{a}}_2 = \begin{pmatrix} 3 \\ -\frac{4}{\sqrt{5}} \end{pmatrix}$$

its norm is $\|\widetilde{\mathbf{a}}_2\| = \sqrt{\frac{61}{5}}$ and the corresponding Householder vector is

$$\widetilde{\mathbf{u}} = \begin{pmatrix} 3+\sqrt{\frac{61}{5}} \\ -\frac{4}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} 6.492 \\ 1.788 \end{pmatrix}$$

Hence its norm square is $\|\widetilde{\mathbf{u}}\|^2 = 45.346$, so

$$H(\widetilde{\mathbf{a}}_2) = I - \frac{2\widetilde{\mathbf{u}}\widetilde{\mathbf{u}}^t}{45.346} = \begin{pmatrix} -0.859 & 0.512 \\ 0.512 & 0.859 \end{pmatrix}$$

From this we form the corresponding 3×3 Householder matrix

$$H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -0.859 & 0.512 \\ 0 & 0.512 & 0.859 \end{pmatrix}$$

and

$$A_2 = H_2 A_1 = \begin{pmatrix} -2.236 & -0.894 & -2.236 \\ 0 & -3.492 & -2.862 \\ 0 & 0 & -0.896 \end{pmatrix}$$

This is the R-matrix in the QR-decomposition. To obtain Q we compute

$$Q := H_1 H_2 = \begin{pmatrix} -.447 & -.458 & -.768 \\ 0 & -.859 & .512 \\ -.894 & .229 & .384 \end{pmatrix}$$

Hence

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 2 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -.447 & -.458 & -.768 \\ 0 & -.859 & .512 \\ -.894 & .229 & .384 \end{pmatrix} \begin{pmatrix} -2.236 & -0.894 & -2.236 \\ 0 & -3.492 & -2.862 \\ 0 & 0 & -0.896 \end{pmatrix}$$

is the QR decomposition. If we insist on positive diagonal elements in R, then we have to multiply both matrices by $-I$:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 2 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} .447 & .458 & .768 \\ 0 & .859 & -.512 \\ .894 & -.229 & -.384 \end{pmatrix} \begin{pmatrix} 2.236 & 0.894 & 2.236 \\ 0 & 3.492 & 2.862 \\ 0 & 0 & 0.896 \end{pmatrix}$$

3.4 Givens rotations

The other possible class of orthogonal transformations which we could use for a stable QR-decomposition is the rotations. Recall that the matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

rotates any vector in the plane counterclockwise by θ. The basic idea is that by an appropriately chosen θ we can always rotate a given vector into a vector whose second entry is zero:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{x^2+y^2} \\ 0 \end{pmatrix}$$

if

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} \qquad \sin\theta = \frac{-y}{\sqrt{x^2+y^2}}$$

In fact it is not necessary to compute the angle θ itself, we always use only its sine and cosine.

We define rotations in $\mathbf{R}^n$ which rotate counterclockwise by θ in a specified coordinate plane by the following $n \times n$ matrix

$$R(i,j,\theta) = \begin{pmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & \ddots & & & & & & \\ & & & \cos\theta & & -\sin\theta & & & \\ & & & & \ddots & & & & \\ & & & \sin\theta & & \cos\theta & & & \\ & & & & & & \ddots & & \\ & & & & & & & 1 & \\ & & & & & & & & 1 \end{pmatrix}$$

where the trigonometric functions are in the i-th and j-th columns and all other entries are zero. These are called **Givens rotations** (sometimes Jacobi rotations). It is clear that $R(i,j,\theta)$ is an orthogonal matrix (CHECK(*)).

3.5 QR factorization with Givens rotations

The key idea is that multiplication of a matrix A by $R(i,j,\theta)$ makes its a_{ji} offdiagonal element zero if θ is chosen appropriately, namely with

$$\cos\theta = \frac{a_{ii}}{\sqrt{a_{ii}^2+a_{ji}^2}} \qquad \sin\theta = \frac{-a_{ji}}{\sqrt{a_{ii}^2+a_{ji}^2}} \tag{3.2}$$

(CHECK(*))

It seems that one can eliminate all offdiagonal nonzero element by a sequence of Givens rotations. But notice that the zero elements may become nonzero again as a result of further rotations. However, if done in a proper order, then at least one can eliminate all nonzero elements below the diagonal. One can follow the same order as in the Gaussian elimination.

Given a matrix A, start with the first column. Take the first nonzero element in the first column below the diagonal. Say, it is in the k-th row. Eliminate it by left multiplication by $R(1,k,\theta)$, where θ is chosen appropriately. Then go to the next nonzero element in the first column, eliminate it etc.

Once the first column has only zeros below the diagonal, go to the second column, and start eliminating below the diagonal with matrices $R(2,k,\theta)$, $k \geq 3$. The key point is that the zeros in the first column remain zeros after these multiplications (notice that $R(i,j,\theta)$ changes only the i-th and j-th columns and rows, hence $R(2,k,\theta)$, $k \geq 3$ does not touch the first column). Then proceed to the third column etc.

Finally you get an upper triangular matrix as a result of many multiplications by Givens matrices from the left, i.e.

$$G_N G_{N-1} \dots G_2 G_1 A = R$$

The maximal number of Givens matrices is $N = (n-1)(n-2)/2$ for an $n \times n$ square matrix A. But then we have the QR-factorization since we can invert all Givens matrices to get

$$A = QR$$

with

$$Q = G_1^t G_2^t \dots G_N^t$$

To see the structure, we show the steps for a 4×3 general matrix, whose generic elements are denoted by $*$:

We start with

$$A = \begin{pmatrix} * & * & * \\ \underline{*} & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$$

and first eliminate the underlined element (if not zero already). For that we use an appropriate

$R(1,2,\theta)$ matrix:

$$G_1 = \begin{pmatrix} c & -s & & \\ s & c & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

to get

$$A_1 = G_1 A = \begin{pmatrix} c & -s & & \\ s & c & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} * & * & * \\ \underline{*} & * & * \\ * & * & * \\ * & * & * \end{pmatrix} = \begin{pmatrix} * & * & * \\ 0 & * & * \\ \underline{*} & * & * \\ * & * & * \end{pmatrix}$$

Here we abbreviated sin and cos by s, c. The choice is, of course

$$c = \frac{a_{11}}{\sqrt{a_{11}^2 + a_{21}^2}} \qquad s = \frac{-a_{21}}{\sqrt{a_{11}^2 + a_{21}^2}}$$

Next we eliminate again the underlined element in A_1 with a new rotation matrix $G_2 = R(1,3,\theta')$, with an appropriate (new) θ:

$$A_2 := G_2 A_1 = \begin{pmatrix} c' & & -s' & \\ & 1 & & \\ s' & & c' & \\ & & & 1 \end{pmatrix} \begin{pmatrix} * & * & * \\ 0 & * & * \\ * & * & * \\ * & * & * \end{pmatrix} = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \\ \underline{*} & * & * \end{pmatrix}$$

Finally we eliminate the last (underlined) entry of the first column by a matrix $G_3 = R(1,4,\theta'')$:

$$A_3 = G_3 A_2 = \begin{pmatrix} c'' & & & -s'' \\ & 1 & & \\ & & 1 & \\ s'' & & & c'' \end{pmatrix} \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \\ * & * & * \end{pmatrix} = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & \underline{*} & * \\ 0 & * & * \end{pmatrix}$$

and we finished with the first column.

Now we continue with the second column by eliminating again the underlined element in A_3 by a matrix $G_4 = R(2,3,\widetilde{\theta})$

$$A_4 = G_4 A_3 = \begin{pmatrix} 1 & & & \\ & \widetilde{c} & -\widetilde{s} & \\ & \widetilde{s} & \widetilde{c} & \\ & & & 1 \end{pmatrix} \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & \underline{*} & * \\ 0 & * & * \end{pmatrix} = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \\ 0 & \underline{*} & * \end{pmatrix}$$

Finally we eliminate the last nonzero element under the diagonal in A_4 by a matrix $G_5 = R(2,4,\widehat{\theta})$

$$A_5 = G_5 A_4 = \begin{pmatrix} 1 & & & \\ & \widehat{c} & & -\widehat{s} \\ & & 1 & \\ & \widehat{s} & & \widehat{c} \end{pmatrix} \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \\ 0 & \underline{*} & * \end{pmatrix} = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix}$$

This last matrix is upper triangular, this will be the R matrix of the factorization.

Summarizing, we have

$$G_5 G_4 G_3 G_2 G_1 A = R$$

hence $A = QR$ with

$$Q = G_1^t G_2^t G_3^t G_4^t G_5^t$$

Problem 3.2 *Find the QR factorization of* $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$ *with Givens rotations.*

SOLUTION: First we aim to kill $a_{21} = 1$. Since $a_{11} = 1$, the corresponding "cosine" and "sine" are

$$c = \frac{1}{\sqrt{1^2+1^2}} = \frac{1}{\sqrt{2}} = .707 \qquad s = \frac{-1}{\sqrt{1^2+1^2}} = -\frac{1}{\sqrt{2}} = -.707$$

and the first rotation matrix is

$$G_1 = \begin{pmatrix} .707 & .707 & 0 \\ -.707 & .707 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$A_1 = G_1 A = \begin{pmatrix} .707 & .707 & 0 \\ -.707 & .707 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1.41 & 2.12 & .707 \\ 0 & -.707 & .707 \\ 2 & 1 & 0 \end{pmatrix}$$

Next we kill the 2 in the last entry of the first column. The corresponding "cosine" and "sine" are

$$c = \frac{1.41}{\sqrt{1.41^2+2^2}} = .577 \qquad s = \frac{-2}{\sqrt{1.41^2+2^2}} = -.816$$

and the second rotation matrix is

$$G_2 = \begin{pmatrix} .577 & 0 & .816 \\ 0 & 1 & 0 \\ -.816 & 0 & .577 \end{pmatrix}$$

and

$$A_2 = G_2 A_1 = \begin{pmatrix} 2.44 & 2.03 & .407 \\ 0 & -.707 & .707 \\ 0 & -1.15 & -.576 \end{pmatrix}$$

Finally we kill the -1.15 with

$$c := \frac{-.707}{\sqrt{(-.707)^2 + (-1.15)^2}} = -.523 \qquad s := \frac{1.15}{\sqrt{(-.707)^2 + (-1.15)^2}} = .851$$

hence

$$G_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -.523 & -.851 \\ 0 & .851 & -.523 \end{pmatrix}$$

and

$$A_3 = G_3 A_2 = \begin{pmatrix} 2.44 & 2.03 & .408 \\ 0 & 1.35 & 1.21 \\ 0 & 0 & .903 \end{pmatrix}$$

This is the R matrix in the QR-decomposition of A.

Finally we collect the Q matrix, we have $G_3 G_2 G_1 A = R$, i.e. $A = QR$ with

$$Q = G_1^t G_2^t G_3^t = \begin{pmatrix} .707 & -.707 & 0 \\ .707 & .707 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} .577 & 0 & -.816 \\ 0 & 1 & 0 \\ .816 & 0 & .577 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -.523 & .851 \\ 0 & -.851 & -.523 \end{pmatrix}$$

$$= \begin{pmatrix} .408 & .86 & -.301 \\ .408 & .123 & .904 \\ .816 & -.492 & -.301 \end{pmatrix}$$

Notice that we needed two Givens matrices (in general $(n-1)$) just to complete the elimination of the first column. The Householder reflection did it in one step. However, computing the Householder matrix is more complicated. More importantly, multiplication of a matrix A by a Householder matrix, changes all its entries, so we really have to compute n^2 new entries (in case of square matrix). Givens rotations leave intact all but two rows and columns. So eventually Givens rotations are not much worse, in fact one can easily check that it requires roughly twice as many elementary operations as Householder.

EXERCISE: Compute the number of elementary operations (addition, multiplication, division, square root) needed to QR-factorize a general $n \times m$ matrix A with the Householder method and with the Givens method.

It is interesting to compare how these two algorithms perform in reality. This can be an interesting computer project. A computer test run on randomly generated matrices reveals that

(i) Householder is faster, especially for larger matrices, but

(ii) Givens is slightly more accurate.

The reason is that Householder is a "greedier" algorithm: it tries to zero more elements at the same time. Hence it is faster, but "lousier". Givens is a slow but more accurate algorithm. However, the error is in fact almost negligible in both cases.

Here are the results of a program by Nolan Leaky. It tests 100 randomly generated matrix (entries are independent random numbers between -1 and 1) for Householder and Givens:

Householder method on 5 by 5 matrix

time: 20.197513ms

max error: 0.000000774860382

avg error: 0.000000297142536

Givens method on 5 by 5 matrix

time: 30.437694ms

max error: 0.000000417232513

avg error: 0.000000139699955

Householder method on 7 by 7 matrix

time: 52.122372ms

max error: 0.000000655651093

avg error: 0.000000336978791

Givens method on 7 by 7 matrix

time: 128.836702ms

max error: 0.000000298023224

avg error: 0.000000183248035