Name:____________________

Test 1
Math 2601 C2
January 22, 2001

Directions : You have 50 minutes to complete all 7 problems on this exam. There are a possible 100 points to be earned on this exam. You may use calculators if you wish. Please be sure to show all pertinent work. *An answer with no work will receive very little credit!* If any portion of the exam is unclear please come to me and I will elaborate provided I can do so without giving away the problem.

(15 points)

1) Compute the following vector operations and sketch the geometry behind the operations.

i) $(2,1)+(1,2)$
ii) $-3(1,1)$
iii) $(2,1,0)\times(-1,1,0)$
iv) $\|\,(2,1,0)\times(-1,1,0)\,\|$
v) $\mathbf{proj}_{\vec{e_2}}(1,2)$

(15 points)

2) Find the equation of a line that passes through the point $P = (1, -1, 2)$ and is parallel to the line $\mathbf{r}(t) = t(3, -1, 1)$

Solution : To find the equation of a line all we need is a point on the line and a vector in the direction of the line. We have the point given to us. Since the line is parallel to $\mathbf{r}(t) = t(3, -1, 1)$ it must also have the same direction vector (up to a scalar multiple). The direction vector for $\mathbf{r}(t) = t(3, -1, 1)$ is the vector $\vec{d} = (3, -1, 1)$. So, the desired line will be $\mathbf{l}(t) = (1, -1, 2) + t(3, -1, 1)$.

(15 points)

3) Find the angle between the planes

$$\mathbb{P}_1 = \{(x,y,z) \mid 5(x-1) - 3(y+2) + 2z = 0\}$$
$$\mathbb{P}_2 = \{(x,y,z) \mid x + 3(y-1) + 2(z+4) = 0\}$$

Solution : We will be looking for an angle θ so that $\cos\theta = |\vec{u}_{\vec{N_1}} \cdot \vec{u}_{\vec{N_2}}|$ where $\vec{u}_{\vec{N_1}}$ is the unit vector in the direction of the normal vector for $\mathbb{P}_1$ and $\vec{u}_{\vec{N_2}}$ is the unit vector in the direction of the normal vector for $\mathbb{P}_2$. Fortunately, the equations given for the planes are written so that we may read of the normal vectors. Indeed, $\vec{N_1} = (5,-3,2)$ and $\vec{N_2} = (1,3,2)$. To find unit vectors with these directions we need only divide each normal vector by its magnitude. That is $\vec{u}_{\vec{N_1}} = \frac{\vec{N_1}}{\|\vec{N_1}\|} = \frac{(5,-3,2)}{\sqrt{5^2+(-3)^2+2^2}} = \frac{1}{\sqrt{38}}(5,-3,2)$. Similarly we see that $\vec{u}_{\vec{N_2}} = \frac{1}{\sqrt{14}}(1,3,2)$. So, we have $\cos\theta = |\vec{u}_{\vec{N_1}} \cdot \vec{u}_{\vec{N_2}}| = |\frac{1}{\sqrt{38}}(5,-3,2) \cdot \frac{1}{\sqrt{14}}(1,3,2)| = |\frac{1}{\sqrt{38}}\frac{1}{\sqrt{14}}||(5,-3,2)\cdot(1,3,2)| = \frac{1}{\sqrt{532}}(5-9+4) = \frac{1}{\sqrt{532}}(0) = 0$. Thus, $\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$ radians.

(15 points)

4) Find the equation for the plane that passes through the points $P_1 = (3, -4, -1)$,$P_2 = (3, 2, 1)$, and $P_3 = (-1, 1, -2)$.

Solution : Recall we only need the normal vector $\vec{N}$ and any point in the plane in order to determine the equation of the plane. We have *three* points in the plane, so we only need to find the normal vector. It would suffice to find two vectors which lie in the plane. Since we could then take their cross product and obtain the normal vector. Consider the vectors $\vec{P_1P_2} = (3, 2, 1) - (3, -4, -1) = (0, 6, 2)$ and $\vec{P_1P_3} = (-1, 1, -2) - (3, -4, -1) = (-4, 5, -1)$. Both of these vectors lie in the plane and so now we can compute the normal vector by taking the cross product $\vec{P_1P_2} \times \vec{P_1P_3} = \vec{N}$. Indeed, we have

$$
\begin{aligned}
&\vec{N} = (0, 6, 2) \times (-4, 5, -1) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 6 & 2 \\ -4 & 5 & -1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 6 & 2 \\ 5 & -1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 0 & 2 \\ -4 & -1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 0 & 6 \\ -4 & 5 \end{vmatrix} \\
&= \mathbf{i}(-6 - 10) - \mathbf{j}(0 + 8) + \mathbf{k}(0 - 30) \\
&= -16\mathbf{i} - 8\mathbf{j} - 30\mathbf{k} = (-16, -8, -30)
\end{aligned}
$$

So, we can now fill in the equation of the plane, $\mathbb{P} = \{(x, y, z) \mid -16(x - 3) - 8(y - (-4)) - 30(z - (-1))\}$. Notice this can be reduced a little to give us $\mathbb{P} = \{(x, y, z) \mid 4(x - 3) + 4(y + 4) + 15(z + 1)\}$. I could also have used either P_2 or P_3 instead of P_1.

(15 points)

5) Are the following sets of vectors linearly dependent or linearly independent? Justify your answer.

i) $\{(1,1,0),(1,0,1),(0,1,1)\}$

ii) $\{(\alpha_1,\alpha_2,\alpha_3),(\beta_1,\beta_2,\beta_3),(\gamma_1,\gamma_2,\gamma_3),(\xi_1,\xi_2,\xi_3)\}$

Solution : i) We need to find constants c_1, c_2, and c_3 so that $c_1(1,1,0)+c_2(1,0,1)+c_3(0,1,1)=0$. So we have the following system of equations:

$$\begin{array}{ccccccc} c_1 & + & c_2 & & & = 0 \\ c_1 & & & + & c_3 & = 0 \\ & & c_2 & + & c_3 & = 0 \end{array}$$

The first equation tells us that $c_1=-c_2$. Then using this information, the second equation reads $-c_2+c_3=0 \Rightarrow c_2=c_3$. Next, using this fact in the third equation we have $c_3+c_3=0 \Rightarrow 2c_3=0 \Rightarrow c_3=0$. Backsolving, we see that $c_1=c_2=c_3=0$. Thus, we deduce that the three vectors are linearly independent.

ii) Trying to use the same tricks from part i) is not the way to approach this problem. We are forced to use a little finesse. Remember the following definition of a basis. A set of vectors $\{\vec{v}_i\}_{i=1}^n$ is a basis for a linear subspace S of $\mathbb{R}^n$ provided it is a *maximal* linear independent set in S. That is, if you add any new vectors, you are no longer linearly independent. With that in mind, notice that $\mathbb{R}^3$ is certainly a linear subspace of itself, as pointed out in class. Also, we know the *standard* basis in $\mathbb{R}^3$ is $\vec{e}_1=(1,0,0)$, $\vec{e}_2=(0,1,0)$, and $\vec{e}_3=(0,0,1)$. Since this is a basis it must be a maximal linearly independent set. So, we know that $dim(\mathbb{R}^3)=3$. So, there can be no linear independent sets of vectors that contain more than three vectors. Notice, if there was a set containing four linearly independent vectors then the collection $\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}$ would not be a maximal set of linear independent vectors, hence not a basis! Certainly a contradiction of all things we have grown to know and love. I.e., the vectors in ii) must be linearly *dependent*.

(15 points)

6) Let V and W be linear subspaces of $\mathbb{R}^n$.

i) Is $V \cup W$ a linear subspace of $\mathbb{R}^n$? Justify your answer.

ii) Is $V \cap W$ a linear subspace of $\mathbb{R}^n$? Justify your answer.

Solution : i) Nope. Consider the linear subspaces $V = \{(x, y) \mid (x, y) = t(1, 0) \text{ for some } t \in \mathbb{R}\}$ and $W = \{(x, y) \mid (x, y) = t(0, 1) \text{ for some } t \in \mathbb{R}\}$. This is a fancy way of saying the x-axis and the y-axis respectively. Now consider the vectors $\vec{e}_1 = (1, 0)$ and $\vec{e}_2 = (0, 1)$. We know that $\vec{e}_1 \in V$ so $\vec{e}_1 \in V \cup W$, likewise $\vec{e}_2 \in W \Rightarrow \vec{e}_2 \in V \cup W$. So, if $V \cup W$ is to be a linear subspace it must be closed under addition. However, $\vec{e}_1 + \vec{e}_2 = (1, 1) \notin V \cup W$. So, we conclude that $V \cup W$ is not a linear subspace in general.

ii) Everything works out fine in this problem. We need to show that $V \cap W$ is closed under addition and scalar multiplication. Let us show that it is closed under addition first. Suppose $\vec{x}$ and $\vec{y}$ are vectors in $V \cap W$. Then, by definition of intersection, we know that $\vec{x}$ and $\vec{y}$ are each in V and W. The fact that we have assumed that V and W are linear subspaces will give us the needed structure. Indeed, since $\vec{x}$ and $\vec{y}$ are in V we know that $\vec{x} + \vec{y} \in V$ since V is a linear subspace and therefore closed under addition. Likewise, since $\vec{x}$ and $\vec{y}$ are each in W we have $\vec{x} + \vec{y} \in W$. Thus we have shown $\vec{x} + \vec{y} \in V \cap W$. It remains to show that $V \cap W$ is closed under scalar multiplication. So, let $\vec{x} \in V \cap W$ and also let α be any scalar. As before we have $\vec{x} \in V$ and $\vec{x} \in W$. So, since V and W are closed under scalar multiplication, $\alpha\vec{x} \in W$ and $\alpha\vec{x} \in W$, so $\alpha\vec{x} \in V \cap W$ as desired.

(10 points)

7) Let V be any line that passes through the origin with direction $\vec{d}$. Let W be a plane that passes through the origin with normal vector $\vec{N}$.
Prove or Disprove : $V \boxtimes W = \{\vec{v} \times \vec{w} \mid \vec{v} \in V, \vec{w} \in W\}$ is a linear subspace of $\mathbb{R}^3$.

Solution : Every good exam ought to have one challenging problem. This is it. It's challenging because there is a lot of potential to get sidetracked with details that are not need to solve this problem. For example, a lot of time can be wasted trying to imagine what this space looks like in general. Is it a line? Is it a plane? Is it Superman? Is it all of $\mathbb{R}^3$? Turns out, knowing the answer to this problem can potentially take a lot of time and it's not really crucial to solving the problem. All we need to do is try to show that the necessary algebraic structure is intact. That is, show that the space is closed under addition and scalar multiplication. Suppose that $\vec{x}, \vec{y} \in V \boxtimes W$. Then, $\vec{x} = \vec{v_1} \times \vec{w}_1$, and $\vec{y} = \vec{v}_2 \times \vec{w}_2$ for some $\vec{v}_1, \vec{v}_2 \in V$ and $\vec{w}_1, \vec{w}_2 \in W$. We can even do a little better than that. Since V is just a line that passes through the origin we know that every vector in V is just some scalar multiple of $\vec{d}$. That is, $\vec{v}_1 = \alpha\vec{d}$ and $\vec{v}_2 = \beta\vec{d}$ for some α and β. So, we *really* have $\vec{x} = (\alpha\vec{d}) \times \vec{w}_1$ and $\vec{y} = (\beta\vec{d}) \times \vec{w}_1$. Now we just need to use the properties of cross product.

$\vec{x} + \vec{y} = (\alpha\vec{d}) \times \vec{w}_1 + (\beta\vec{d}) \times \vec{w}_2$
$= \vec{d} \times (\alpha\vec{w}_1) + \vec{d} \times (\beta\vec{w}_2)$
$= \vec{d} \times (\alpha\vec{w}_1 + \beta\vec{w}_2)$

So, notice that $\vec{d} \in V$ and $(\alpha\vec{w}_1 + \beta\vec{w}_2) \in W$, thus their cross product is an element of $V \boxtimes W$ as desired. It remains to show that $V \boxtimes W$ is closed under scalar multiplication. Let $\vec{x} = (\alpha\vec{d} \times \vec{w}) \in V \boxtimes W$ and ξ be any scalar. Then we have $\xi\vec{x} = \xi(\alpha\vec{d} \times \vec{w}) = (\xi\alpha\vec{d}) \times \vec{w}$. Since V is a linear subspace we know $(\xi\alpha\vec{d}) \in V$ and so $(\xi\alpha\vec{d}) \times \vec{w} \in V \boxtimes W$. We conclude that $V \boxtimes W$ is a linear subspace of $\mathbb{R}^3$.