Name:____________________

Test 2
Math 2601 C2
February 16, 2001

Directions : You have 50 minutes to complete all 4 problems on this exam. There are a possible 100 points to be earned on this exam; each problem is worth 25 points. **You may *not* use a calculator**. Please be sure to show all pertinent work. *An answer with no work will receive very little credit!* If any portion of the exam is unclear please come to me and I will elaborate provided I can do so without giving away the problem.

1) Let A be the matrix representation of a linear map $T : V \longrightarrow W$, where V and W are vector spaces and,

$$A = \begin{pmatrix} 1 & -2 & 3 & 4 \\ 2 & -3 & 4 & 7 \\ -3 & -4 & 6 & 8 \end{pmatrix}$$

i) Find $dim(V)$ and $dim(W)$.

ii) Find $Ker(A)$.

iii) Find $Im(A)$.

Solution :

i) A is a 3×4 matrix. So, it must eat vectors in $\mathbb{R}^4$ and spit out vectors in $\mathbb{R}^3$. Thus $dim(V) = 4$ and $dim(W) = 3$.

ii) We only need to row reduce to find $Ker(A)$. So,

$$\begin{pmatrix} 1 & -2 & 3 & 4 \\ 2 & -3 & 4 & 7 \\ -3 & -4 & 6 & 8 \end{pmatrix}$$

$$(R_2 = R_2 + (-2)R_1) \begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & -2 & -1 \\ -3 & -4 & 6 & 8 \end{pmatrix}$$

$$(R_3 = R_3 + (3)R_1) \begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & -10 & 15 & 20 \end{pmatrix}$$

$$(R_3 = R_3 + (10)R_2) \begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & -5 & 10 \end{pmatrix}$$

$$(R_3 = -\frac{1}{5}R_3)\begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

$$(R_2 = R_2 + (2)R_3)\begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

$$(R_1 = R_1 + (-3)R_3)\begin{pmatrix} 1 & -2 & 0 & 10 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

$$(R_1 = R_1 + (2)R_2)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

We have shown that if $A\vec{x} = \vec{0}$ then $\vec{x} = \begin{pmatrix} 0 \\ 5t \\ 2t \\ t \end{pmatrix}$. Thus $Ker(A) = \textbf{span}\left\{\begin{pmatrix} 0 \\ 5 \\ 2 \\ 1 \end{pmatrix}\right\}$.

iii) We just need to look at the work we have done previously to determine $Im(A)$. We see that the first three columns of the row-reduced matrix are each pivotal. Thus the image is spanned by the first three columns of the original matrix. I.e. $Im(A) = \textbf{span}\left\{\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \\ -4 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}\right\}$.

2) Prove or Disprove:
If $T : \mathcal{P}_2 \longrightarrow \mathcal{P}_3$ is the map defined by,

$$T(p(x)) = \left[\int_0^x p(t)dt\right] + 1$$

then, T is linear.
(If you are going to prove the statement, you must use the definition of linearity. If you are going to disprove the statement, you must come up with a counter-example that violates the definition of linearity).

Solution : Recall the definition of linearity.

Definition : A map $T : V \longrightarrow W$, where V and W are vector spaces, is linear if the following two conditions hold,

i) $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ for all $\vec{x}, \vec{y} \in V$.

ii) $T(\alpha\vec{x}) = \alpha T(\vec{x})$ for all $\vec{x} \in V$ and all scalars α.

In our case, T is a map from the vector space $\mathcal{P}_2$ into the vector space $\mathcal{P}_3$. So, our vectors are polynomials of degree at most two.

T is not linear. The easy check is that it fails the condition $T(\alpha p(x)) = \alpha T(p(x))$ when $\alpha = 0$. Indeed, $T(0p(x)) = T(0) = \int_0^x 0dt + 1 = 0 + 1 = 1$, but $0T(p(x)) = 0\left[\int_0^x p(t)dt + 1\right] = 0$. So, we have shown $T(\alpha p(x)) \neq \alpha T(p(x))$ in general and so T is not linear. You could have used the condition $T(p(x) + q(x)) = T(p(x)) + T(q(x))$ to derive a contradiction as well. Take your two favorite polynomials in $\mathcal{P}_2$, let's use $p(x) = 1 + x$ and $q(x) = 1 + x^2$. Then notice,

$$\begin{aligned} T(p(x) + q(x)) &= T((1 + x) + (1 + x^2)) \\ &= T(2 + x + x^2) \\ &= \int_0^x 2 + t + t^2 dt + 1 \\ &= 2x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + 1 \end{aligned}$$

But,

$$\begin{aligned} T(p(x)) + T(q(x)) &= T(1 + x) + T(1 + x^2) \\ &= \left[\int_0^x 1 + tdt + 1\right] + \left[\int_0^x 1 + t^2 dt + 1\right] \\ &= \left[x + \frac{1}{2}x^2 + 1\right] + \left[x + \frac{1}{3}x^3 + 1\right] \\ &= 2x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + 2 \end{aligned}$$

Notice that these two quantities are not equal, so again we deduce that T is not linear. Some of you may be bothered that I chose specific polynolmials to check linearity. Remember you can only do this if you are trying to provide a counter example. Since, the definition of linearity says that these conditions must hold for any polynomials $p(x)$ and $q(x)$ and all scalars α. I have shown that the definition fails when I choose $p(x) = 1 + x$ and $q(x) = 1 + x^2$. Thus the conditions don't hold for *all* polynomials, which is necessary for linearity.

3) Let $\triangle$: $\mathcal{P}_2 \longrightarrow \mathcal{P}_3$ be defined by, $\triangle\,(p(x)) = \int_0^x p(t)dt$. Let $\triangledown : \mathcal{P}_3 \longrightarrow \mathcal{P}_2$ be defined by, $\triangledown(q(x)) = q'(x)$, where $q'(x)$ denotes the usual derivative of $q(x)$. Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be the standard bases for $\mathcal{P}_2$ and $\mathcal{P}_3$ respectively.

i) Find ${}_{\mathfrak{B}_2}M^{\triangle}_{\mathfrak{B}_1}$ and ${}_{\mathfrak{B}_1}M^{\triangledown}_{\mathfrak{B}_2}$.

ii) What is the map $\triangledown \circ \triangle$?

iii) Prove your assertion in part ii).

Solution :

$$ {}_{\mathfrak{B}_2}M^{\triangle}_{\mathfrak{B}_1} = \begin{bmatrix} | & | & | \\ \Psi_{\mathfrak{B}_2}(\triangle\,(1)) & \Psi_{\mathfrak{B}_2}(\triangle\,(x)) & \Psi_{\mathfrak{B}_2}(\triangle\,(x^2)) \\ | & | & | \end{bmatrix} $$

So, let's compute the columns.

$$ \Psi_{\mathfrak{B}_2}(\triangle\,(1)) = \Psi_{\mathfrak{B}_2}(\int_0^x 1dt) = \Psi_{\mathfrak{B}_2}(x) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} $$

$$ \Psi_{\mathfrak{B}_2}(\triangle\,(x)) = \Psi_{\mathfrak{B}_2}(\int_0^x tdt) = \Psi_{\mathfrak{B}_2}(\frac{1}{2}x^2) = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} $$

$$ \Psi_{\mathfrak{B}_2}(\triangle\,(x^2)) = \Psi_{\mathfrak{B}_2}(\int_0^x t^2dt) = \Psi_{\mathfrak{B}_2}(\frac{1}{3}x^3) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{3} \end{pmatrix} $$

So,

$$ {}_{\mathfrak{B}_2}M^{\triangle}_{\mathfrak{B}_1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} $$

Similarly,

$$ {}_{\mathfrak{B}_1}M^{\triangledown}_{\mathfrak{B}_2} = \begin{bmatrix} | & | & | & | \\ \Psi_{\mathfrak{B}_1}(\triangledown(1)) & \Psi_{\mathfrak{B}_1}(\triangledown(x)) & \Psi_{\mathfrak{B}_1}(\triangledown(x^2)) & \Psi_{\mathfrak{B}_1}(\triangledown(x^3)) \\ | & | & | & | \end{bmatrix} $$

Again, we compute the columns.

$$ \Psi_{\mathfrak{B}_1}(\triangledown(1)) = \Psi_{\mathfrak{B}_1}(\frac{d}{dx}1) = \Psi_{\mathfrak{B}_1}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} $$

$$\Psi_{\mathfrak{B}_1}(\nabla(x)) = \Psi_{\mathfrak{B}_1}(\frac{d}{dx}x) = \Psi_{\mathfrak{B}_1}(1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Psi_{\mathfrak{B}_1}(\nabla(x^2)) = \Psi_{\mathfrak{B}_1}(\frac{d}{dx}x^2) = \Psi_{\mathfrak{B}_1}(2x) = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

$$\Psi_{\mathfrak{B}_1}(\nabla(x^3)) = \Psi_{\mathfrak{B}_1}(\frac{d}{dx}x^3) = \Psi_{\mathfrak{B}_1}(3x^2) = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

So,

$$_{\mathfrak{B}_2}M_{\mathfrak{B}_1}^{\triangle} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

ii) This composition is the identity map from $\mathcal{P}_2$ into itself.

iii) To prove this we only need to find a representation for the composition $\nabla \circ \triangle$. This can be done via matrix multiplication. That is,

$$_{\mathfrak{B}_1}M_{\mathfrak{B}_1}^{\nabla\circ\triangle} = \left[{}_{\mathfrak{B}_1}M_{\mathfrak{B}_2}^{\nabla}\right]\left[\mathfrak{B}_2 M_{\mathfrak{B}_1}^{\triangle}\right]$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So, we see that the composition gives us the 3×3 identity matrix, thus proving our assertion.

4) Verify that the following matrix is invertible. (Hint. There are many ways to compute a determinent.)

$$\begin{pmatrix} 1 & 35 & 24 & -12 & 17 \\ 0 & -11 & 17 & 1 & \frac{1}{3} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 52 & 135 & 0 & 0 \\ 0 & 6 & -25 & 0 & \frac{1}{135} \end{pmatrix}$$

Solution : You guessed it. We will take a determinent. Notice that there are a whole bunch of zeros in this matrix. The goal will be to expand along a row or column that has the most zero terms in it. So, it looks like a good place to start will be to expand along either the third row or the first column, since they each only have one nonzero term. Then we will try to find the row or column in the resulting submatrix which has the most zeros etc... I will first expand along the first column. Let us begin,

$$det \begin{pmatrix} 1 & 35 & 24 & -12 & 17 \\ 0 & -11 & 17 & 1 & \frac{1}{3} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 52 & 135 & 0 & 0 \\ 0 & 6 & -25 & 0 & \frac{1}{135} \end{pmatrix}$$

$$= 1(-1)^{1+1} det \begin{pmatrix} -11 & 17 & 1 & \frac{1}{3} \\ 1 & 0 & 0 & 0 \\ 52 & 135 & 0 & 0 \\ 6 & -25 & 0 & \frac{1}{135} \end{pmatrix}$$

Next, expand the submatrix along the second row

$$= 1 \left[1(-1)^{2+1} det \begin{pmatrix} 17 & 1 & \frac{1}{3} \\ 135 & 0 & 0 \\ -25 & 0 & \frac{1}{135} \end{pmatrix} \right]$$

Then, expand along the second column.

$$= 1 \left[(-1) \left[1(-1)^{1+2} det \begin{pmatrix} 135 & 0 \\ -25 & \frac{1}{135} \end{pmatrix} \right] \right]$$

Now, we only have a 2×2 matrix to deal with.

$$= 1 \left[(-1) \left[(-1) \left[(135)(\frac{1}{135}) - (0)(-25) \right] \right] \right]$$

$$= 1(-1)(-1)(1) = 1$$

All that work for 1, sigh... Well $1 \neq 0$ so the matrix has full rank and is therefore invertible.