Name:____________________

Test 3
Math 2601 C2
March 19, 2001

Directions : You have 50 minutes to complete all 4 problems on this exam. There are a possible 100 points to be earned on this exam; each problem is worth 25 points. You may use calculators. Please be sure to show all pertinent work. *An answer with no work will receive very little credit!* If any portion of the exam is unclear please come to me and I will elaborate provided I can do so without giving away the problem.

1) Let $\vec{x} = \begin{pmatrix} 2 \\ -4 \\ 4 \end{pmatrix}$, and let $\vec{y} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$.

i) Find a Householder reflection matrix H so that $H\vec{x} = \begin{pmatrix} c_1 \\ 0 \\ 0 \end{pmatrix}$,

OR

find a Givens rotation matrix G so that $G\vec{y} = \begin{pmatrix} c_2 \\ 0 \\ 0 \end{pmatrix}$.

ii) Find a matrices Q_1,R_1,Q_2, and R_2 so that $\vec{x} = Q_1R_1$ and $\vec{y} = Q_2R_2$, where Q_1 has orthonormal columns, Q_2 has orthonormal columns, and R_1,R_2 are upper triangular.

Solution :

First let us find the Householder reflection matrix H. We know that $H = I - 2\vec{u}\vec{u}^t$, where $\vec{u}$ is the unit vector in the same direction as the vector $\begin{pmatrix} 2 + sign(2) \parallel \vec{x} \parallel \\ -4 \\ 4 \end{pmatrix}$. As if by magic, the value of $\parallel \vec{x} \parallel$ is nice, indeed $\parallel \vec{x} \parallel = 6$. So, we have the vector $\begin{pmatrix} 8 \\ -4 \\ 4 \end{pmatrix}$ and we need to find a unit vector in this direction. That is $\vec{u} = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$. So, we have some computation to do,

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\frac{1}{6} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \end{pmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

You should check and verify that $H\vec{x} = \begin{pmatrix} -6 \\ 0 \end{pmatrix}$.

Now lets compute a Givens rotation matrix G so that $G\vec{y}$ is upper triangular. Fortunately (thanks Chad!) we only need to compute one matrix, since the vector $\vec{y}$ has only one term to be eliminated. The desired Givens rotation will have the form,

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Where,

$$\cos\theta = \frac{3}{\sqrt{3^2+4^2}} = \frac{3}{5}$$
$$\sin\theta = -\frac{4}{\sqrt{3^2+4^2}} = -\frac{4}{5}$$

Wow! No radicals in either part! What are the odds? Anyway, we see that,

$$G = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

You should check and verify that $G\vec{y} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}$.

Now, lets solve part **ii)**. There are many possible solutions to this problem, I will list several below.

$$Q_1 = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}^T \qquad R_1 = \begin{bmatrix} -6 \\ 0 \\ 0 \end{bmatrix}$$

$$Q_1' = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix} \qquad R_1' = [6]$$

$$Q_2 = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \qquad R_2 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$$

$$Q_2' = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \\ 0 \end{bmatrix} \qquad R_2' = [5]$$

2) Solve the following system of differential equations.

$$\mathbb{S} = \left\{ \begin{array}{l} x_1'(t) = x_1(t) - 4x_2(t) \\ x_2'(t) = -4x_1(t) + x2(t) \end{array} \right\}$$

Solution :

We know the solution will have the form $\vec{x}(t) = e^{At}\vec{x}(0)$, where $A = \begin{bmatrix} 1 & -4 \\ -4 & 1 \end{bmatrix}$, and since there is no given initial condition we have in general, $\vec{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$.

So, in order to find e^{At} we will need the eigenvalues and eigenvectors. To find the eigenvalues we need only find the roots of the characteristic polynomial $\det(A - \lambda I)$. Indeed,

$$\begin{vmatrix} 1-\lambda & -4 \\ -4 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 16$$

$$= \lambda^2 - 2\lambda - 15 = (\lambda - 5)(\lambda + 3)$$

So, we have found the eigenvalues to be $\lambda_1 = 5$ and $\lambda_2 = -3$. We now need to find the eigenvectors for these eigenvalues. Let's start with $\lambda_1 = 5$ first. We know an eigenvector for this eigenvalue is an element in the kernel of $(A - \lambda_1 I)$. So, we only need to row reduce.

$$\begin{bmatrix} 1-\lambda_1 & -4 & | & 0 \\ -4 & 1-\lambda_1 & | & 0 \end{bmatrix} = \begin{bmatrix} -4 & -4 & | & 0 \\ -4 & -4 & | & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

So, an eigenvector $\vec{v}_1$ for the eigenvalue $\lambda_1 = 5$ is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. To find the remaining eigenvector we need to row reduce the following,

$$\begin{bmatrix} 1-\lambda_2 & -4 & | & 0 \\ -4 & 1-\lambda_2 & | & 0 \end{bmatrix} = \begin{bmatrix} 4 & -4 & | & 0 \\ -4 & 4 & | & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

So, an eigenvector $\vec{v}_2$ for the eigenvalue $\lambda_2 = -3$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Define $V := \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ and $D := \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix}$. Then we know our solution is of the form $Ve^{Dt}V^{-1}$. Let us compute V^{-1}.

$$\begin{bmatrix} -1 & 1 & | & 1 & 0 \\ 1 & 1 & | & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 & | & 1 & 0 \\ 0 & 2 & | & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 & | & 1 & 0 \\ 0 & 1 & | & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & | & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & | & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

So, we have our solution,

$$\vec{x}(t) = \begin{bmatrix} -1 & 1 \\ \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{5t} & 0 \\ \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{pmatrix} x_1(0) \\ \\ x_2(0) \end{pmatrix}$$

3) Consider the matrix $B = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}$. Find $\| B \|$

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Solution :

Recall, we are using the operator norm defined by $\| B \| = \begin{matrix} max \\ \vec{v} \in \mathbb{R}^3 \\ \vec{v} \neq \vec{0} \end{matrix} \frac{\|B\vec{v}\|}{\|\vec{v}\|}$. Since the matrix we were given is symmetric, we know that the norm will just be the largest eigenvalue (in absolute value). So, we need to compute another characteristic polynomial.

$$\begin{vmatrix} 1-\lambda & 1 & -1 \\ 1 & -\lambda & 1 \\ -1 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)\left[-\lambda(1-\lambda)-1\right] - (1-\lambda+1) - (1-\lambda)$$

$$= (1-\lambda)\left[\lambda^2 - \lambda - 1\right] + 2\lambda - 3$$

$$= (\lambda^2 - \lambda - 1 - \lambda^3 + \lambda^2 + \lambda) + 2\lambda - 3$$

$$= -\lambda^3 + 2\lambda^2 + 2\lambda - 4$$

We know, by examining the constant term and the leading coefficient, that any rational root will be on of the following, $\pm\frac{4}{1}, \pm\frac{2}{1}, \pm 11$. Indeed, the rational root for this polynomial is 2. The polynomial factors to $-1(\lambda-2)(\lambda-\sqrt{2})(\lambda+\sqrt{2})$. Since 2 is the largest in absolute value, we have $\| B \| = 2$.

4) Let $C = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. Find a solution $\vec{x}^*$, of minimum length, to the system of equations $C\vec{x} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

Solution :

This is an underdetermined system, and so a minimal solution will be $\vec{x}^* = P_r\vec{x}_0$, where P_r is the projection onto the row space of C and $\vec{x}_0$ is *any* solution to $C\vec{x} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$. Let us first find a suitable $\vec{x}_0$. Since the matrix is already in reduced row echelon form (how nice!) we see that all solutions will be of the form,

$$\vec{x} = \begin{pmatrix} 3 - s - t \\ s \\ 0t \end{pmatrix}$$

Since we are allowed to choose $\vec{x}_0$ to be any solution why not try to make it look as simple as possible. Say,

$$\vec{x}_0 = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

That is, let $s = t = 0$.

To find P_r we need to find a QR decomposition for C^T. Since the rows of C are orthogonal already, we know the columns of C^T are orthogonal, so we only need to normalize them. No Gram-Schmidt! (Again, how nice!) We have,

$$C^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{3}} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix}$$

Thus,

$$P_r = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{3}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{3}} & 0 \end{bmatrix}^T$$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{3}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$$

Then, our solution is

$$\vec{x}^* = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Let's just see what happened if I chose to use the general form of the solution. I should arrive at the same $\vec{x}^*$, which is amazing.

$$P_r\vec{x} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \begin{pmatrix} 3 - s - t \\ s \\ 0 \\ t \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{s}{3} - \frac{t}{3} + \frac{s}{3} + \frac{t}{3} \\ 1 - \frac{s}{3} - \frac{t}{3} + \frac{s}{3} + \frac{t}{3} \\ 0 + 0 + 0 + 0 \\ 1 - \frac{s}{3} - \frac{t}{3} + \frac{s}{3} + \frac{t}{3} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Yup, that's amazing!