Differentiation Formulas

(1) $\frac{d}{dt}\left(\vec{f}+\vec{g}\right)=\frac{d\vec{f}}{dt}+\frac{d\vec{g}}{dt}$
(2) $\frac{d}{dt}\left(\boldsymbol{a}\vec{f}\right)=\boldsymbol{a}\frac{d\vec{f}}{dt}$
(3) $\frac{d}{dt}\left(u\vec{f}\right)=u\frac{d\vec{f}}{dt}+\frac{du}{dt}\vec{f}$
(4) $\frac{d}{dt}\left(\vec{f}\cdot\vec{g}\right)=\left(\vec{f}\cdot\frac{d\vec{g}}{dt}\right)+\left(\frac{d\vec{f}}{dt}\cdot\vec{g}\right)$
(5) $\frac{d}{dt}\left(\vec{f}\times\vec{g}\right)=\left(\vec{f}\times\frac{d\vec{g}}{dt}\right)+\left(\frac{d\vec{f}}{dt}\times\vec{g}\right)$
(6) $\frac{d\vec{f}}{dt}=\frac{d\vec{f}}{du}\frac{du}{dt}$

Example: Let $\vec{r}$ be a differentiable vector function of t and set $r=\|\vec{r}\|$. Show that where $r\neq 0$

$$\frac{d}{dt}\left(\frac{\vec{r}}{r}\right)=\frac{1}{r^3}\left[\left(\vec{r}\times\frac{d\vec{r}}{dt}\right)\times\vec{r}\right]$$

Solution, for the most part we just fall down the rabbit hole of what we have to do. That is, start on the left side, compute the derivative, and see what happens.

$\frac{d}{dt}\left(\frac{\vec{r}}{r}\right)=\frac{1}{r}\frac{d\vec{r}}{dt}-\frac{1}{r^2}\frac{dr}{dt}\vec{r}$ by (3) in the table above.

$\frac{1}{r}\frac{d\vec{r}}{dt}-\frac{1}{r^2}\frac{dr}{dt}\vec{r}=\frac{1}{r^3}\left[r^2\frac{d\vec{r}}{dt}-r\frac{dr}{dt}\vec{r}\right]$ by factoring out $\frac{1}{r^3}$.

$\frac{1}{r^3}\left[r^2\frac{d\vec{r}}{dt}-r\frac{dr}{dt}\vec{r}\right]=\frac{1}{r^3}\left[(\vec{r}\cdot\vec{r})\frac{d\vec{r}}{dt}-r\frac{dr}{dt}\vec{r}\right]$ since $r=\|\vec{r}\|\Rightarrow r^2=\|\vec{r}\|^2=\vec{r}\cdot\vec{r}$.

$\frac{1}{r^3}\left[(\vec{r}\cdot\vec{r})\frac{d\vec{r}}{dt}-r\frac{dr}{dt}\vec{r}\right]=\frac{1}{r^3}\left[(\vec{r}\cdot\vec{r})\frac{d\vec{r}}{dt}-\left(\vec{r}\cdot\frac{d\vec{r}}{dt}\right)\vec{r}\right]$. This step is true for the following reasons.

$r=\|\vec{r}\|\Rightarrow r^2=\|\vec{r}\|^2=\vec{r}\cdot\vec{r}$.

So, $r^2=\vec{r}\cdot\vec{r}\iff 2r\frac{dr}{dt}=\vec{r}\cdot\frac{d\vec{r}}{dt}+\frac{d\vec{r}}{dt}\cdot\vec{r}\iff 2r\frac{dr}{dt}=2\vec{r}\frac{d\vec{r}}{dt}\iff r\frac{dr}{dt}=\vec{r}\frac{d\vec{r}}{dt}$.

$\frac{1}{r^3}\left[(\vec{r}\cdot\vec{r})\frac{d\vec{r}}{dt}-\left(\vec{r}\cdot\frac{d\vec{r}}{dt}\right)\vec{r}\right]=\frac{1}{r^3}\left[\left(\vec{r}\times\frac{d\vec{r}}{dt}\right)\times\vec{r}\right]$. This follows from the identity, $(c\cdot a)b-(c\cdot b)a=(a\times b)\times c$ with the following identifications,

$$a=\vec{r}$$
$$b=\frac{d\vec{r}}{dt}$$
$$c=\vec{r}$$