It is absolutely essential to this course that you feel comfortable calculating limits. The calculation of certain limits will allow for us to use some powerful machinery, which will determine convergence or divergence of infinite series. With this is mind it seems appropriate to offer a ***very*** watered down review of limits. If you have any difficulty with the following problems it is strongly suggested that you go back and review the material.

Calculate the following limits:

1) $\lim\limits_{x \to -\infty}\left[\dfrac{2-x}{\sqrt{7+6x^2}}\right]$

2) $\lim\limits_{x \to -\infty}\left[\dfrac{\sqrt{3x^4+x}}{x^2-8}\right]$

3) $\lim\limits_{x \to 1^+}\left[\dfrac{x^4-1}{x-1}\right]$

4) $\lim\limits_{x \to 4^-}\left[\dfrac{3-x}{x^2-2x-8}\right]$

5) $\lim\limits_{x \to 3}\left[\dfrac{x}{x-3}\right]$

6) $\lim\limits_{x \to \infty}\left[\dfrac{6-x^3}{7x^3+3}\right]$

7) $\lim\limits_{x \to \infty}(\sqrt{x^2+3}-x)$

8) $\lim\limits_{x \to \infty}(\sqrt{x^2+5x}-x)$

9) $\lim\limits_{x \to \infty}\left[\dfrac{c_0+c_1x+\cdots+c_nx^n}{d_0+d_1x+\cdots+d_mx^m}\right]$, where $c_n \neq 0$ and $d_m \neq 0$

10) $\lim\limits_{q \to 0}\left[\dfrac{\sin(q)}{q}\right]$

11) $\lim\limits_{x \to 0^-}\left[\dfrac{\sin(x)}{|x|}\right]$

12) $\lim\limits_{l \to 0}\left[\dfrac{\sin(3l)}{\sin(5l)}\right]$

13) $\lim\limits_{x \to 0}\left[\dfrac{x}{\cos(x)}\right]$

14) $\lim\limits_{x \to 0}\left[\dfrac{\tan(7x)}{\sin(3x)}\right]$

15) $\lim\limits_{h \to 0}\left[\dfrac{\sin(h)}{\sqrt{x}(\sec^2(x))}\right]$

Solutions:

1) $\lim_{x \to -\infty}\left[\frac{2-x}{\sqrt{7+6x^2}}\right] = \lim_{x \to -\infty}\left[\frac{(2-x)\left(\frac{1}{|x|}\right)}{(\sqrt{7+6x^2})\left(\frac{1}{|x|}\right)}\right] = \lim_{x \to -\infty}\left[\frac{(2-x)\left(\frac{1}{-x}\right)}{(\sqrt{7+6x^2})\left(\frac{1}{\sqrt{x^2}}\right)}\right] =$

$= \lim_{x \to -\infty}\left[\frac{1-\frac{2}{x}}{\sqrt{\frac{7}{x^2}+6}}\right] = \frac{1}{\sqrt{6}}$

2) $\lim_{x \to -\infty}\left[\frac{\sqrt{3x^4+x}}{x^2-8}\right] = \lim_{x \to -\infty}\left[\frac{\sqrt{3x^4+x}\left(\frac{1}{|x^2|}\right)}{(x^2-8)\left(\frac{1}{|x^2|}\right)}\right] = \lim_{x \to -\infty}\left[\frac{\sqrt{\frac{3x^4}{x^4}+\frac{x}{x^4}}}{\left(\frac{x^2}{x^2}-\frac{8}{x^2}\right)}\right] =$

$\lim_{x \to -\infty}\left[\frac{\sqrt{3+\frac{1}{x^3}}}{\left(1-\frac{8}{x^2}\right)}\right] = \sqrt{3}$

3) $\lim_{x \to 1^+}\left[\frac{x^4-1}{x-1}\right] = \lim_{x \to 1^+}\left[\frac{(x^2+1)(x^2-1)}{x-1}\right] = \lim_{x \to 1^+}\left[\frac{(x^2+1)(x-1)(x+1)}{x-1}\right] =$

$= \lim_{x \to 1^+}[(x^2+1)(x+1)] = 4$

4) $\lim_{x \to 4^-}\left[\frac{3-x}{x^2-2x-8}\right] = \lim_{x \to 4^-}\left[\frac{3-x}{(x-4)(x+2)}\right] = +\infty$

5) $\lim_{x \to 3}\left[\frac{x}{x-3}\right]$ Does Not Exist. Since, $\lim_{x \to 3^-}\left[\frac{x}{x-3}\right] = -\infty$ and $\lim_{x \to 3^+}\left[\frac{x}{x-3}\right] = +\infty$

6) $$\lim_{x \to \infty}\left[\frac{6-x^3}{7x^3+3}\right]=\lim_{x \to \infty}\left[\frac{\frac{6}{x^3}-\frac{x^3}{x^3}}{\frac{7x^3}{x^3}+\frac{3}{x^3}}\right]=\lim_{x \to \infty}\left[\frac{\frac{6}{x^3}-1}{7+\frac{3}{x^3}}\right]=\frac{-1}{7}$$

7) $$\lim_{x \to \infty}(\sqrt{x^2+3}-x)=\lim_{x \to \infty}\left[\frac{(\sqrt{x^2+3}-x)(\sqrt{x^2+3}+x)}{(\sqrt{x^2+3}+x)}\right]=$$

$$=\lim_{x \to \infty}\left[\frac{(x^2+3-x^2)}{(\sqrt{x^2+3}+x)}\right]=\lim_{x \to \infty}\left[\frac{3}{(\sqrt{x^2+3}+x)}\right]=0$$

8) $$\lim_{x \to \infty}(\sqrt{x^2+5x}-x)=\lim_{x \to \infty}\left[\frac{(\sqrt{x^2+5x}-x)(\sqrt{x^2+5x}+x)}{(\sqrt{x^2+5x}+x)}\right]=$$

$$=\lim_{x \to \infty}\left[\frac{(x^2+5x-x^2)}{(\sqrt{x^2+5x}+x)}\right]=\lim_{x \to \infty}\left[\frac{5x}{(\sqrt{x^2+5x}+x)}\right]=\lim_{x \to \infty}\left[\frac{5x\left(\frac{1}{|x|}\right)}{(\sqrt{x^2+5x}+x)\left(\frac{1}{|x|}\right)}\right]=$$

$$=\lim_{x \to \infty}\left[\frac{\left(\frac{5x}{x}\right)}{(\sqrt{x^2+5x}+x)\left(\frac{1}{\sqrt{x^2}}\right)}\right]=\lim_{x \to \infty}\left[\frac{5}{(\sqrt{1+\frac{5}{x}}+1)}\right]=\frac{5}{2}$$

9) $\lim_{x\to\infty}\left[\dfrac{c_0+c_1x+\cdots+c_nx^n}{d_0+d_1x+\cdots+d_mx^m}\right]$, where $c_n\neq 0$ and $d_m\neq 0$. There are three cases

I. If n>m then, $\lim_{x\to\infty}\left[\dfrac{c_0+c_1x+\cdots+c_nx^n}{d_0+d_1x+\cdots+d_mx^m}\right]=\lim_{x\to\infty}\left[\dfrac{\frac{c_0}{x^m}+\frac{c_1x}{x^m}+\cdots+\frac{c_nx^n}{x^m}}{\frac{d_0}{x^m}+\frac{d_1x}{x^m}+\cdots+\frac{d_mx^m}{x^m}}\right]=$

$$=\lim_{x\to\infty}\left[\dfrac{\frac{c_0}{x^m}+\frac{c_1}{x^{m-1}}+\cdots+c_nx^{n-m}}{\frac{d_0}{x^m}+\frac{d_1}{x^{m-1}}+\cdots+d_m}\right]=+\infty$$

II. If n<m then, $\lim_{x\to\infty}\left[\dfrac{c_0+c_1x+\cdots+c_nx^n}{d_0+d_1x+\cdots+d_mx^m}\right]=\lim_{x\to\infty}\left[\dfrac{\frac{c_0}{x^n}+\frac{c_1x}{x^n}+\cdots+\frac{c_nx^n}{x^n}}{\frac{d_0}{x^n}+\frac{d_1x}{x^n}+\cdots+\frac{d_mx^m}{x^n}}\right]=$

$$=\lim_{x\to\infty}\left[\dfrac{\frac{c_0}{x^n}+\frac{c_1}{x^{n-1}}+\cdots+c_n}{\frac{d_0}{x^n}+\frac{d_1}{x^{n-1}}+\cdots+d_mx^{m-n}}\right]=0$$

III. If n=m then, $\lim_{x\to\infty}\left[\dfrac{c_0+c_1x+\cdots+c_nx^n}{d_0+d_1x+\cdots+d_mx^m}\right]=\lim_{x\to\infty}\left[\dfrac{c_0+c_1x+\cdots+c_nx^n}{d_0+d_1x+\cdots+d_nx^n}\right]=$

$$=\lim_{x\to\infty}\left[\dfrac{c_0+c_1x+\cdots+c_nx^n}{d_0+d_1x+\cdots+d_nx^n}\right]=\lim_{x\to\infty}\left[\dfrac{\frac{c_0}{x^n}+\frac{c_1x}{x^n}+\cdots+\frac{c_nx^n}{x^n}}{\frac{d_0}{x^n}+\frac{d_1x}{x^n}+\cdots+\frac{d_nx^n}{x^n}}\right]=$$

$$=\lim_{x\to\infty}\left[\dfrac{\frac{c_0}{x^n}+\frac{c_1}{x^{n-1}}+\cdots+c_n}{\frac{d_0}{x^n}+\frac{d_1}{x^{n-1}}+\cdots+d_n}\right]=\frac{c_n}{d_n}$$

10) $\lim_{q \to 0}\left[\frac{\sin(q)}{q}\right]$ =1. This was a given fact up until now. In this class we will use L'Hopital's rule to prove that is the correct solution.

11) $\lim_{x \to 0^-}\left[\frac{\sin(x)}{|x|}\right] = \lim_{x \to 0^-}\left[\frac{\sin(x)}{-x}\right] = -1$

12) $\lim_{l \to 0}\left[\frac{\sin(3l)}{\sin(5l)}\right] = \lim_{l \to 0}\left[\frac{\frac{3l\sin(3l)}{3l}}{\frac{5l\sin(5l)}{5l}}\right] = \lim_{l \to 0}\left[\frac{\frac{3\sin(3l)}{3l}}{\frac{5\sin(5l)}{5l}}\right] = \left[\frac{3\left(\lim_{l \to 0}\frac{\sin(3l)}{3l}\right)}{5\left(\lim_{l \to 0}\frac{\sin(5l)}{5l}\right)}\right] =$

$= \left[\frac{3(1)}{5(1)}\right] = \frac{3}{5}$

13) $\lim_{x \to 0}\left[\frac{x}{\cos(x)}\right] = \frac{\lim_{x \to 0}(x)}{\lim_{x \to 0}[\cos(x)]} = \frac{0}{1} = 0$

14) $\lim_{x \to 0}\left[\frac{\tan(7x)}{\sin(3x)}\right] = \lim_{x \to 0}\left[\frac{\sin(7x)}{\cos(7x)\sin(3x)}\right] = \lim_{x \to 0}\left[\frac{\frac{7x\sin(7x)}{7x}}{\cos(7x)\frac{3x\sin(3x)}{3x}}\right] =$

$= \lim_{x \to 0}\left[\frac{\frac{7\sin(7x)}{7x}}{\cos(7x)\frac{3\sin(3x)}{3x}}\right] = \frac{\lim_{x \to 0}\left[\frac{7\sin(7x)}{7x}\right]}{\lim_{x \to 0}\left[\cos(7x)\frac{3\sin(3x)}{3x}\right]} =$

$$= \frac{\lim\limits_{x \to 0}\left[\frac{7\sin(7x)}{7x}\right]}{\left(\lim\limits_{x \to 0}[\cos(7x)]\right)\left(\lim\limits_{x \to 0}\left[\frac{3\sin(3x)}{3x}\right]\right)} = \frac{7}{(1)(3)} = \frac{7}{3}$$

15) $\lim\limits_{h \to 0}\left[\frac{\sin(h)}{\sqrt{x}(\sec^2(x))}\right] = \frac{1}{\sqrt{x}(\sec^2(x))}\lim\limits_{h \to 0}[\sin(h)] = \frac{1}{\sqrt{x}(\sec^2(x))}(0) = 0$