RESEARCH STATEMENT
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1. Introduction and Background

The types of problems I enjoy working on tend to be a combination of geometry and topology. More specifically I have been studying knot theory related to energies and invariants. Some examples of knot invariants include crossing number, bridge number, and genus each of which are computed as the minimum value of a functional over an entire isotopy class. The knot energies are similarly defined. If we denote the space of knotted curves as $\mathcal{K}$, then a knot energy is a functional $f : \mathcal{K} \longrightarrow \mathbb{R}$, and we are interested in a knot that minimizes an energy for its isotopy class. One can think of the knot's energy value as a measure of a curve's resistance to being in a particular configuration.

One such energy is the ropelength of a curve, which is defined as the quotient of length by the radius of the largest embedded tubular neighborhood around the curve. The ropelength measures the amount of rope necessary to tie a given knot. Bounds on ropelength have been found in terms of different topological invariants, and it is this aspect of the ropelength problem that I enjoy working on. Another knot energy is Gromov's distortion which is the maximum over all pairs of points on the knot of the quotient of arclength by spatial distance. The closer a curve's distortion is to 1, the closer the curve is to being a geodesic. Techniques similar to those used to find ropelength bounds have, as of yet, not been very successful at finding distortion bounds. Instead, in my thesis I describe some properties of knots that distortion minimizers must possess. Aside from this work, I have also participated as a co-leader in research groups for undergraduates. The idea of these groups is to give undergraduate students enough background in a subject, in our case ropelength, so that they get experience working on open problems in mathematics. In the future I would like to continue working on the distortion problem by applying different analytical techniques such as mathematical signal analysis and the calculus of variations.

1.1. **Ropelength.** The ropelength of a knot is defined as follows:

Definition 1. *Let $\gamma : [0, 1] \longrightarrow \mathbb{R}^3$ be a simple closed curve. We can define the ropelength of γ to be the quotient of length by the radius of the largest embedded tubular neighborhood around the curve. We also define the ropelength of an isotopy class to be the minimum ropelength of all members of the isotopy class.*

Finding the ropelength of any particular isotopy class is difficult. In fact, the only known minimizer for knots is the round circle which minimizes the ropelength for the class of the unknot. Aside from that example, the best that has been done is to achieve lower and upper bounds for ropelength in terms of other topological invariants. The goal of [2], a paper that was the result of a research group led by Jason Cantarella, is to achieve such bounds in terms of the crossing number of a knot. The crossing number of a knot is defined in terms of crossing numbers of knot projections. A knot projection is simply a planar projection where we keep track of where strands of a knot cross over or under other strands of the knot. We only insist that to have a well defined projection that the crossings be transverse and that there is only one crossing at a given point in the projection. The crossing number for that projection is then simply the number of crossings that occur. The crossing number of the isotopy class of a knot γ, denoted $c(\gamma)$, is the minimum crossing number that can be achieved by generic projections of knots in the isotopy class. A result from [2], a joint paper with Jason Cantarella, Xander Faber, and myself, is the following:

Theorem 1. *If L is a non-split link, then*

$$\textit{Ropelength}(L) \leq 1.64c(L)^2 + 7.69c(L) + 6.74,$$

In particular, this bound holds for prime links.

To do this, we embed the knots as an arc-presentation which is defined as follows:

Definition 2. *An* arc-presentation *of a link L is an embedding of L in a finite collection of α open half-planes arrayed around a common axis, or binding, so that the intersection of L with each half-plane is a single simple arc. The number of half-planes α is called the* arc-index *of the arc-presentation. The minimal arc-index over all arc-presentations of a link L is an invariant of the knot type.*

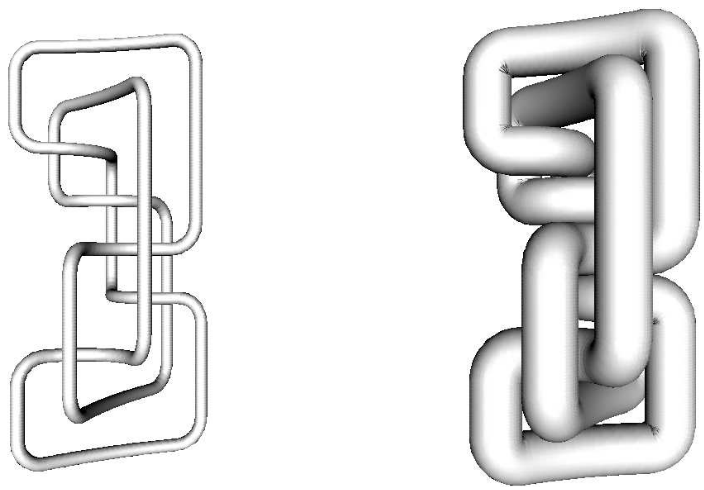

FIGURE 1. Here we see the 7_1 knot drawn in the arc-presentation style described in [2]. On the left we see a thin version of the knot which enables us to better see details obfuscated in the thick knot on the right. It is the thick knot whose ropelength we are interested in.

The idea of the paper was to fix the thickness of the knot and bound its length. We took an arc-presentation of the knot, which lined up all of the crossings on the z-axis, and studied the length required to connect the crossings.

2. RESEARCH OBJECTIVES

2.1. **Distortion.** My thesis work involves the distortion of knotted curves. We can define this energy as follows

Definition 3. *The distortion of a closed curve $\gamma : [0,1] \longrightarrow \mathbb{R}^3$ parametrized by arclength is given by*

$$\delta(\gamma) = \sup_{s \neq t} \frac{\min\{|s-t|, 1-|s-t|\}}{\|\gamma(s) - \gamma(t)\|} \tag{1}$$

and the distortion of an entire isotopy class $[\gamma]$ is

$$\delta([\gamma]) = \inf_{\gamma \in [\gamma]} \delta(\gamma). \tag{2}$$

In [4] M. Gromov wonders whether or not every knot class has a representative whose distortion is less than 100. Many people have worked very hard to obtain an answer to this question by trying to find lower bounds for distortion in terms of known topological invariants, such as crossing number. However, the distortion energy has proven to be an exceptionally elusive quantity when an attempt is made to minimize it. A reason this is to be expected is because the distortion is a supremum over pairs of points. If we have a pair of points that realize the distortion and then we perturb the curve, it may be (and is in fact likely) a completely different pair of points will realize the distortion. There is very little known control on the location of the pairs of points that realize the distortion throughout an isotopy. Another difficulty is that there is no known way to easily minimize distortion by minimizing another simpler functional with respect to some constraint. One would like to extend recent techniques used to obtain bounds on ropelength. For example, when dealing with ropelength one can fix the thickness of the knot and work on minimizing the length required to complete the knot. This type of approach will not work with distortion since neither the numerator or denominator of the distortion quotient can be treated as a constraint. It seems to be the case so far that *any* attempt at minimizing the distortion of knots meets with similar snags. In [5] Jun O'Hara devised a family of knot energies defined as,

$$e_j^p[\gamma] := \frac{1}{j}\left[\iint \left(\frac{1}{\|\gamma(s)-\gamma(t)\|} - \frac{1}{d(s,t)^j}\right)^p \mathrm{d}s\ \mathrm{d}t\right]^{\frac{1}{p}}$$

where $d(s,t)$ denotes the shortest distance along the curve from s to t, whose limit as $p \to \infty$ and $j \to 0$ is the natural log of distortion. He was successful in constructing the converging family of energies, which are interesting to study in their own right, and did obtain lower bounds for each energy e_j^p in terms of crossing number. Unfortunately, the limit of these upper bounds as $p \to \infty$ approached zero for fixed crossing number. So, the lower bound became useless and did not provide a nontrivial lower bound on distortion.

To my knowledge, the best that has been done at getting any type of foothold on the distortion problem is recent work in [3]. In this paper Denne and Sullivan obtain a bound on distortion using the existence of what are known as essential arcs, which are present in all nontrivial knots. Their theorem states that any nontrivial finite total curvature curve has distortion at least 3.9945.

My approach to this problem is somewhat different. Instead of trying to construct bounds for distortion I study the geometric properties of distortion minimizers. If one wishes to prove a lower bound on distortion for curves in a given knot type, it suffices to prove that bound for distortion minimizing curves in that knot type. Therefore any geometric information known about these distortion minimizers is helpful in obtaining such bounds. We start with some definitions.

Definition 4. *We say a closed curve γ is d-admissable provided γ has finite total curvature and $\delta(\gamma) \leq d$.*

Next,

Definition 5. *We say a closed curve γ is safely scaled provided for all (s,t) if the distortion quotient evaluated at the points (s,t) is larger than $\delta([\gamma])$, then $\|\gamma(s)-\gamma(t)\| \geq 1$.*

The idea is to think of a particular knot class and it's minimum distortion d. Then if we consider all of the d-admissable curves in that particular knot type, we know we will only be dealing with knots that achieve the minimum. We care that the knots be safely scaled because we are going to want to consider knots of minimum length. Caution is required since distortion is scale invariant. Length minimizers will not exist unless we fix scale. The fact that a knot is safely scaled, we hope, will prevent a knot from changing isotopy type when its length is being minimized. My theorem is then the following,

Theorem 2. *Let γ be a d-admissable safely scaled knot of minimum length. Then any open interval on γ either has total curvature zero, or contains an endpoint of a $\delta([\gamma])$ realizing chord.*

The reason we are concerned with curves of minimum length is due to the following heuristic. Suppose that a curve contained an arc of positive curvature that does not contain either member of a pair which realizes the distortion of the isotopy class. Then we can shorten the curve in this region *without* introducing new points that realize the distortion $\delta([\gamma])$. In fact, if all pairs which realize distortion measure their arclength through this region, the distortion will actually be *decreased*. But if even one pair doesn't measure arclength through this section, distortion will be unchanged. The hypothesis is then the assumption that all these shortenings have been done already.

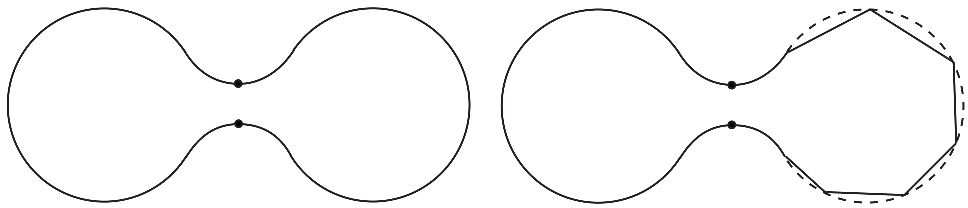

FIGURE 2. On the left we see a curve where the distortion is controlled by the pair of points displayed. Suppose that the curve is symmetric about a vertical line through these points. We can replace the right circular segment with a polygonal approximation and this will then decrease the distortion of the curve since the shortest distance used in computing arclength will necessarily travel through the shorter polygonal region.

The proof of the theorem involves showing that given an interval of nonzero curvature, we can decrease the length without affecting the distortion between pairs of points very much. Then show that if there is an interval that does not contain the endpoints of any distortion realizing chord, then distortion quotient is bounded away from the distortion of the curve along this interval. Once these two propositions have been established we can then prove the theorem as follows. Assume that γ is a curve that satisfies the hypotheses of the theorem and yet contains an interval of non-zero curvature which also does not contain any points that realize the distortion. Then, using the previous propositions, we know that we can decrease the length of this interval without changing the curves distortion. This contradicts the fact that the curve had minimum length.

2.2. **A Simpler Family of Functionals.** We would like to directly apply the calculus of variations to distortion to compute Euler-Lagrange equations. However, the distortion is a sup function and so we cannot easily apply the calculus of variations. A similar problem to investigate involves a simpler energy functional and it's different L_p variants. In [1], Abrams, Cantarella, Fu, Ghomi, and Howard discuss a generalization of O'Hara's energy by considering the functional

$$f[\gamma] := \iint F(\|\gamma(s) - \gamma(t)\|, d(s,t)) \mathrm{d}s \, \mathrm{d}t,$$

where $d(s,t)$ again denotes the shortest distance along the curve from s to t and $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

They discuss replacing the function F with $F_p(x,y) = -x^p$ for various values of p. This new energy corresponds to the average chord length of the chord γ measured in the L^p norm. They conjecture that there is a critical exponent p^* so that the F_p energies are minimized by round circles exactly when $p \leq p^*$. They also note that the critical exponent p^* (if it exists) obeys $2 \leq p^* < 3.5721$. This analysis leads to several open problems, a couple of which I plan to work on. These problems are

(1) Find the critical exponent p^*, if it exists.

(2) Describe the shape of the minimizers for the renormalization energies based on $F_p = -x^p$ for $p > p^*$.

It is my hope that computing Euler-Lagrange equations for these different energies will shed some light on each of these questions as well as give me a much deeper understanding of the calculus of variations to be used on distortion.

2.3. **Software.** In addition to the theoretical side of this work, I have also developed software that will lower distortion of knotted polygons. The software searches for the pairs of points that realize the distortion and attempts to perturb them so that the distortion decreases. This simulated annealing then approximates curves that are local minima for the distortion functional.

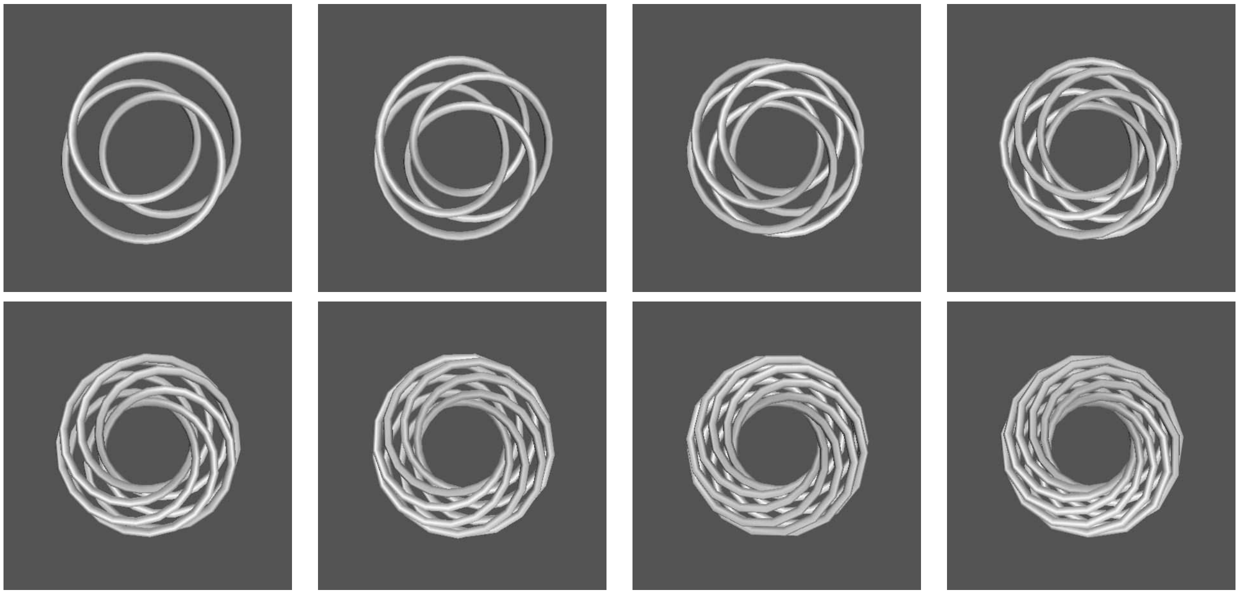

FIGURE 3. Here we see a collection of $(n, n-1)$ torus knots for $n = 3, 4, \ldots, 10$ parametrized by the equation $(3 + \cos(2\pi(n-1)t)\cos(2\pi nt), 3 + \cos(2\pi(n-1)t)\sin(2\pi nt), \sin(2\pi(n-1)t))$ approximated by polygons consisting of 150 edges.

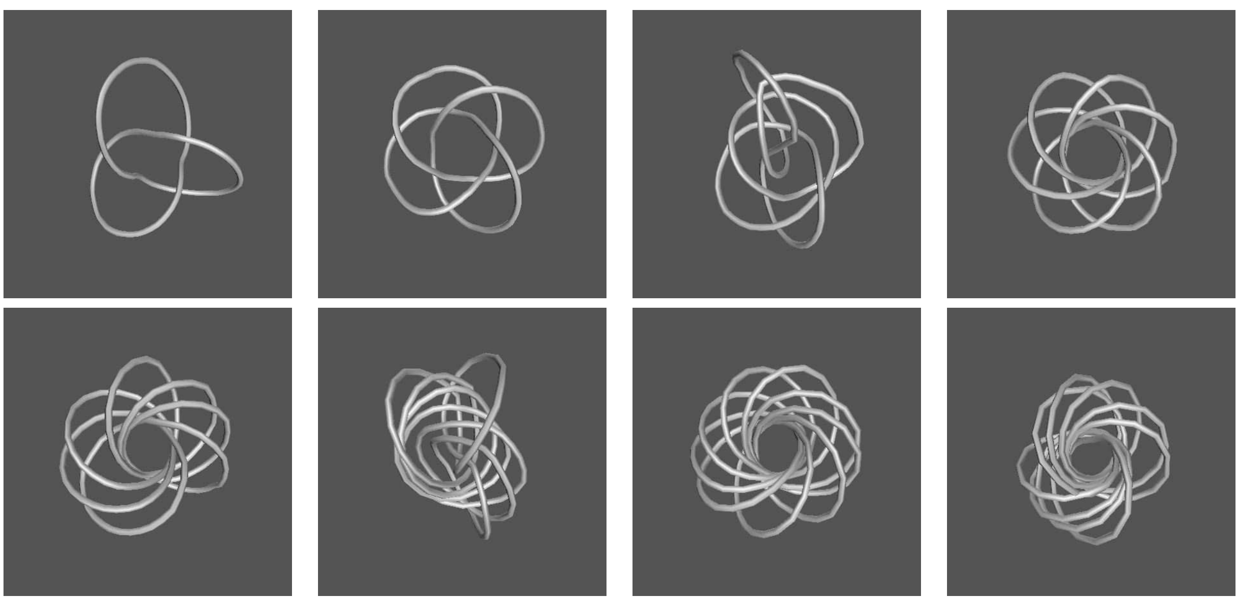

FIGURE 4. We see the output after running my software on the $(n, n-1)$ torus knots in the previous figure. It is interesting to note that the knots appear to have moved from the $(n, n-1)$ embedding to the $(n-1, n)$ embedding.

While running these computations the locations of the distortion realizing chords is computed. While the number of distortion critical chords is low, they are hopping around as the annealing takes place. The value of the distortion change is also very small after running for a while and control of the distortion alternates between several pairs of vertices. So, the computer code is beginning to exhibit the properties of my theorem and we can expect distortion realizing chords at most (if not all) vertices if we were to run the code long enough. It was my thought that the value of the distortion for each of these knot types should increase as n increases. This appears to be the case with the data that have been collected so far as the below graph illustrates.

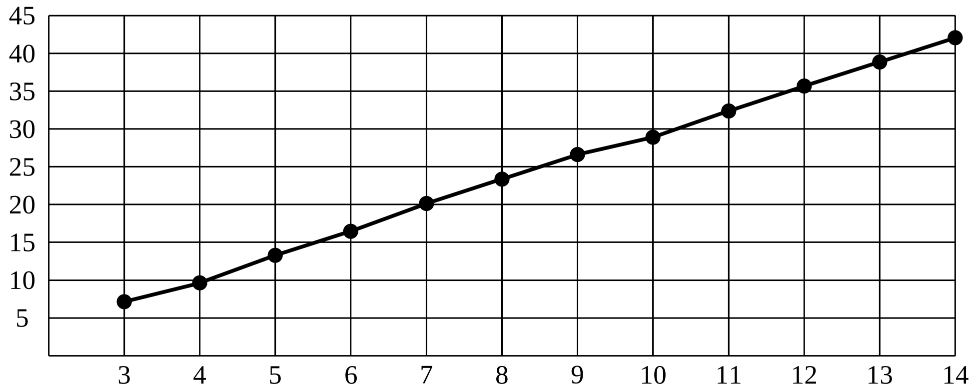

FIGURE 5. The horizontal axis corresponds to twelve $(n, n-1)$ torus knots that were output from the code. The vertical axis corresponds to the value of the distortion of the knots.

3. CAREER DEVELOPMENTS

The University of Georgia was fortunate enough to have received a VIGRE grant from the NSF. Due to this grant my advisor has been able to lead research groups during the academic year as well as REU's (Research Experience for Undergraduates) during the summer months. I have been involved in all of these groups for four years. The groups have focused on answering some questions about the ropelength problem. During the summer I was the co-leader and taught the participants the material they needed to know to be effective researchers. I covered topics in knot theory including basic definitions and an introduction to such knot invariants as the HOMFLY, Jones, and Alexander polynomials. The students seemed to enjoy their work and I am very eager to run similar research groups as soon as I am able.

I am also looking forward to obtaining teaching experience at a different location. This will allow for me to experience a different emphasis on topics taught so that I may further modify and improve my teaching style. In addition, I will also be exposed to other faculty members that can provide new insight to research that I am developing.

[1] Aaron Abrams, Jason Cantarella, Joseph H. G. Fu, Mohammad Ghomi, and Ralph Howard. Circles minimize most knot energies. *Topology*, 42(2):381-394, 2003.

[2] Jason Cantarella, X.W. Faber, and Chad A.S. Mullikin. Upper bounds for ropelength as a function of crossing number. *Topology and its Applications,* 135:253-264, 2003.

[3] Elizabeth Denne and John M. Sullivan. The distortion of a knotted curve. *arXiv:math.GT/0409438*, 2004.

[4] Mikhael Gromov. Filling Riemannian manifolds. *J. Differential Geom.,* 18(1)1-147, 1983.

[5] Jun O'Hara. Energy functionals of knots. In *Topology Hawaii (Honolulu, HI, 1990),* pages 201-214. World Sci. Publishing, River Edge, NJ, 1992.