SKETCHING EXPONENTIAL FUNCTIONS

Here we are interested in examining various changes that can be made to an exponential function of the form $f(x) = a^x$ (where I will first assume $a > 1$) and what affect these changes have on the resulting graph. There are a few fundamental changes we can make.

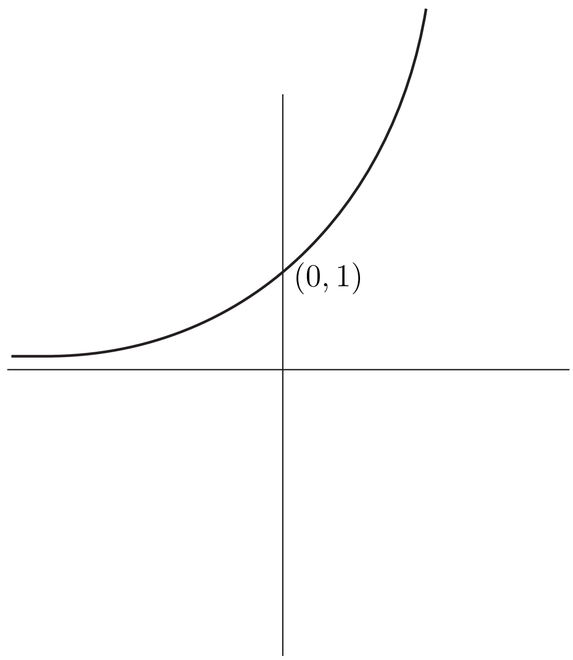
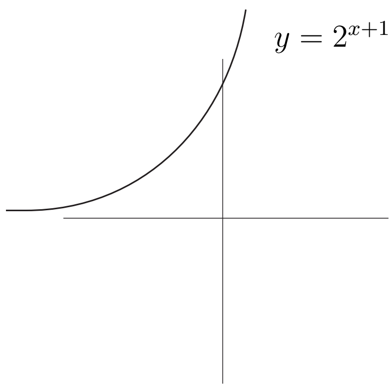
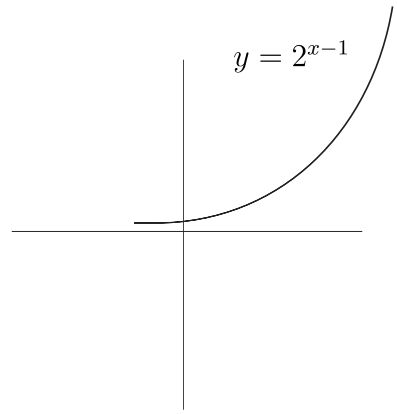


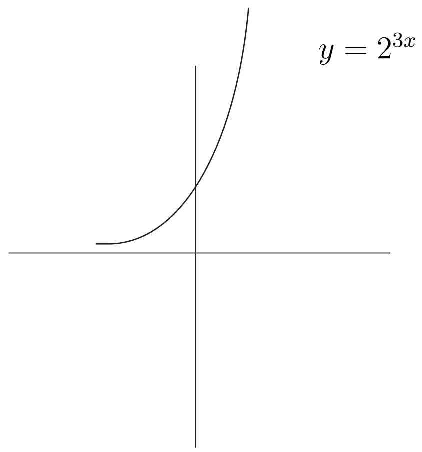
FIGURE 1. The graph of the function $f(x) = 2^x$.

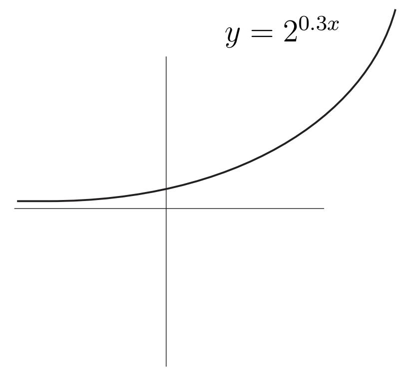
(1) Changes to the independent variable x by addition and subtraction. For example $f(x) = 2^{x+1}$ will shift the entire graph to the left by 1 unit. In general the graph of $y = 2^{x+c}$ will be a shift the of the graph $y = 2^x$ by c units to the left if $c > 0$ and to the right if $c < 0$.

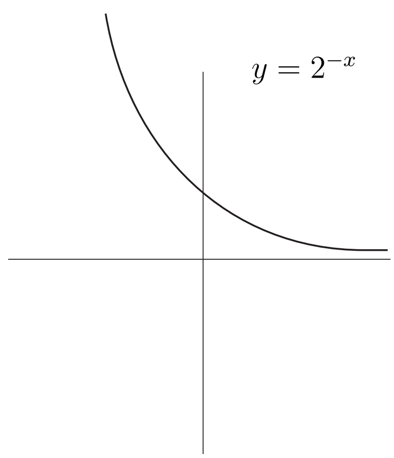




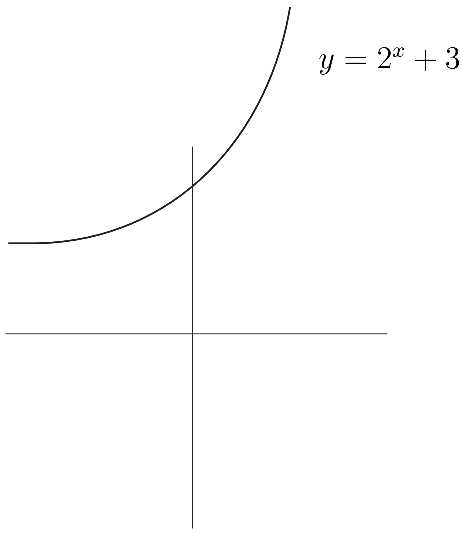
(2) Changes to the independent variable x by constant multiplication. The graph of $y = 2^{cx}$ (with $c > 0$) can be imagined by taking the graph of $y = 2^x$ and stretching it out (or compressing it) left and right (imagine stretching the x-axis) by a factor of c. If $c > 1$ we compress the x-axis and if $0 < c < 1$ we stretch the x-axis. If $c < 0$, then in addition to the stretch and compression, the graph will be flipped around the y-axis.

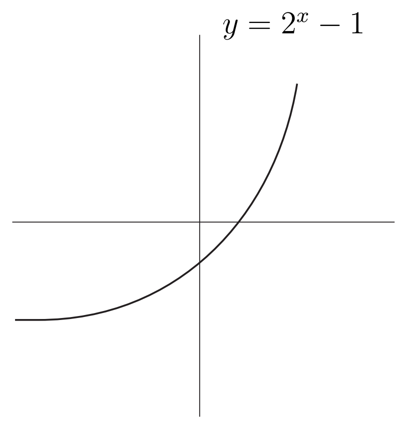






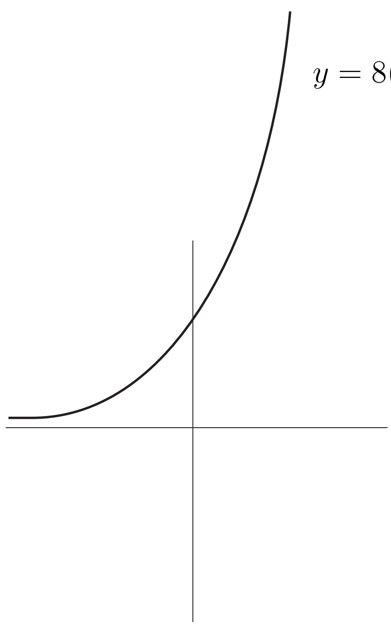
(3) Changes to the dependent variable y by addition and subtraction. The graph of $y = 2^x + c$ will be a shift the of the graph $y = 2^x$ by c units up if $c > 0$ and c units down if $c < 0$.

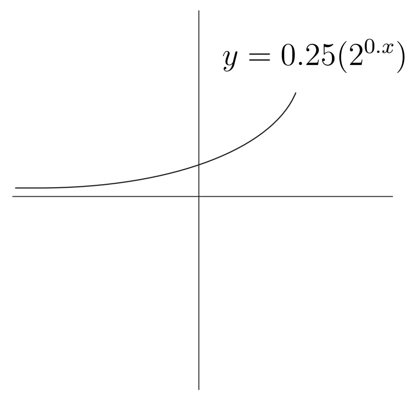




(4) Changes to the dependent variable y by multiplication. This is the example that is a little mysterious since we are dealing with the exponential function. The graph of the function $y = 2^x$ is the same as the graph of $y = 2^x$ where all of the y-values have been multiplied by the constant c. This is similar to the stretching and compressing of the x-axis mentioned previously. Instead of the stretching and compression happening on the x-axis it now happens on the y-axis. Notice also that if $c < 0$ this will flip the graph about the x-axis. Here comes the weird part. We can also realize the change of the graph as a shift to the left or the right by using the laws of exponents. For example, consider the function $g(x) = 8\,(2^x)$. How does $g(x)$ differ from the function $f(x) = 2^x$. Well, the obvious answer is that all of the y-values of $g(x)$ are the same as all the y-values of $f(x)$ once they have been multiplied by the constant 8. This results in stretching the y-axis. Sure enough the point $(0, 1)$ on the graph of $y = f(x)$ got stretched up to the point $(0, 8)$. But, check this out.

$$g(x) = 8(2^x) = (2^3)(2^x) = 2^{x+3}.$$



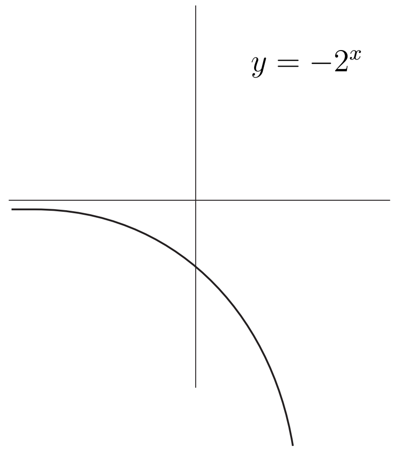


Rewriting the function $g(x)$ as above we see that the graph of $g(x) = 8\,(2^x)$ is the same as the graph of $f(x) = 2^x$ shifted to the left 3 units. So, which interpretation is correct? They both are. For those who love generality (who doesn't?!) look at the following. First assume that $c > 0$ and a is a positive real number not equal to 1. In this case, there exists some real number power p so that $a^p = c$ and we can write

$$ca^x = (a^p)a^x = a^{x+p}.$$

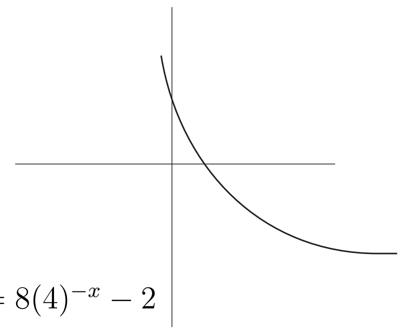
This shows that constant multiplication of *any* exponential function $f(x) = a^x$ (with a a positive number other than 1) by a positive number c to get the function $g(x) = ca^x$ can be realized as a shift to the left or right as well as a stretch or compression of the y-axis.

Last, if we multiply by -1, then the graph is flipped about the x-axis.



Example : Sketch the graph of $f(x) = 8(4)^{-x} - 2$.

Solution : First we need to see if we can tell what has been modified. We have multiplied the y-values by 8, negated the x-values, and subtracted 2 from the y-values. The subtract of the 2 occurs after everything else has been done, so we will focus on the $8(4)^{-x}$ term first. The 8 will stretch the graph along the y-axis by a factor of 8, or using the new improved techniques, we see that this may also be interpreted as a right shift by 3/2 since $8(4)^{-x} = (4^{3/2})(4)^{-x} = 4^{-(x-3/2)}$. The fact that the x term has been multiplied by a -1 means that the graph will be reflected around the y-axis. Finally, by subtracting 2 from everything we shift the graph down by 2 units.



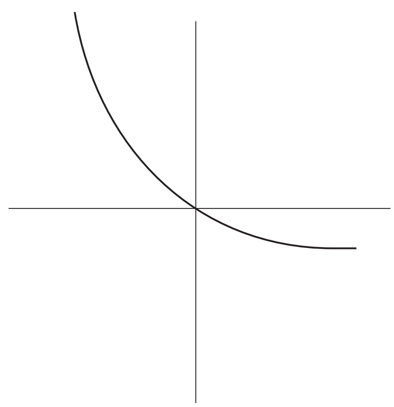
$y = 8(4)^{-x} - 2$

Example : Sketch the graph of the function $f(x) = (1/2)^x - 1$.

Solution : In this write up, we have not yet talked about functions of this form. Recall that at the beginning I assumed that we would be talking about functions of the form $f(x) = a^x$ where $a > 1$. Well, what happens if $0 < a < 1$? This about taking a number in between 0 and 1 and multiplying it by itself. Does it get bigger or smaller? It get smaller right? For example $(0.1)(0.1) = 0.01$, $(0.1)(0.1)(0.1) = 0.001$, etc... Why is this? Well, this example can illustrate what is going on. The graph of the unction $f(x) = (1/2)^x - 1$ will be the same as the graph of the function $g(x) = (1/2)^x$ after we shift $g(x)$ down 1 unit. Now, check out how we can rewrite the function $g(x)$.

$$g(x) = (\frac{1}{2})^x = \frac{1^x}{2^x} = \frac{1}{2^x} = 2^{-x}.$$

We see that the graph of $g(x)$ is then the same as the graph of $y = 2^x$ after we flip it around the y-axis. So, to sketch the graph of $y = f(x)$ we start with the function we know and love $y = 2^x$, flip it around the y-axis and then shift it down 1 unit.



Long story short, if presented with a function of the form $f(x) = (1/a)^x$, rewrite the function as $f(x) = a^{-x}$ and go from there.