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Test 2
Summer 2007
MTH121 Section 01
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Directions : You have 60 minutes to complete all 5 problems on this exam. There are a possible 100 points to be earned. You may not use your book, notes, or any graphing/programmable calculator. Please be sure to show all pertinent work. *A correct answer with no work will receive very little credit!* If any portion of the exam is unclear please come to me and I will elaborate provided I can do so without giving away the problem.

1. (20 points)
The **wandering zombie** is the curve given by the equation

$$y^4 - 2y^3 - x^3 - y^2 + 3x^2 + 2y - 2x = 0.$$

Use implicit differentiation to find the slope of the line tangent to the wandering zombie at the point $(1, 2)$.

Solution :

$$\frac{d}{dx}\left[y^4 - 2y^3 - x^3 - y^2 + 3x^2 + 2y - 2x\right] = \frac{d}{dx}0$$

$$4y^3\frac{dy}{dx} - 6y^2\frac{dy}{dx} - 3x^2\frac{dx}{dx} - 2y\frac{dy}{dx} + 6x\frac{dx}{dx} + 2\frac{dy}{dx} - 2\frac{dx}{dx} = 0$$

$$4y^3\frac{dy}{dx} - 6y^2\frac{dy}{dx} - 3x^2 - 2y\frac{dy}{dx} + 6x + 2\frac{dy}{dx} - 2 = 0$$

$$\frac{dy}{dx}\left(4y^3 - 6y^2 - 2y + 2\right) = 3x^2 - 6x + 2$$

$$\frac{dy}{dx} = \frac{3x^2 - 6x + 2}{4y^3 - 6y^2 - 2y + 2}.$$

We have computed the slope of the tangent line at any point (x, y) on the curve. The slope of the tangent line at the point $(1, 2)$ is:

$$\frac{3(1)^2 - 6(1) + 2}{4(2)^3 - 6(2)^2 - 2(2) + 2} = -\frac{1}{6}.$$

2. (20 points)
A particle moves along the curve $y = \sqrt{1+x^3}$. As it reaches the point $(2,3)$, the y-coordinate is increasing at a rate of 4 cm/s. How fast is the x-coordinate of the point changing at that instant?

Solution : We are given $dy/dt = 4$ at the point $(2,3)$ and we are asked to find dx/dt. To do this, we compute a derivative of the given equation and substitute all of the known values, then solve for dx/dt.

$$\frac{d}{dt}y = \frac{d}{dt}\sqrt{1+x^3}$$

$$\frac{dy}{dt} = \frac{1}{2}\left(1+x^3\right)^{-1/2}(3x^2\frac{dx}{dt}$$

$$4 = \frac{1}{2}\left(1+2^3\right)^{-1/2}(3(2)^2)\frac{dx}{dt}$$

$$4 = \frac{1}{2}\left(\frac{1}{\sqrt{9}}\right)(12)\frac{dx}{dt}$$

$$4 = \frac{12}{6}\frac{dx}{dt}$$

$$4 = 2\frac{dx}{dt}$$

$$2 = \frac{dx}{dt}.$$

3. (20 points)
   Use a linear approximation to estimate $\sqrt{170}$.

   Solution : Let $f(x) = \sqrt{x}$ and observe that we wish to approximate $f(17)$. We will use a linear approximation $\ell(x) = f'(a)(x-a) + f(a)$ at a point a close to 170 since when a is close to 170 $\ell(170) \approx f(170)$. We need to choose a value for a for which we can compute $f(a)$ and $f'(a)$ as both are needed in the equation for $\ell(x)$. Let $a = 169$. Then a is relatively close to 170 , $f(169) = \sqrt{169} = 13$, and $f'(169) = 1/(2\sqrt{169}) = 1/26$. So, the equation for the tangent line is

$$\ell(x) = \frac{1}{26}(x - 169) + 13,$$

   and

$$\sqrt{170} = f(170) \approx \ell(170) = \frac{1}{26}(170 - 169) + 13 = \frac{339}{26}.$$

4. (20 points)
Find the maximum and minimum value of the function on the indicated interval.

$$f(x) = \frac{x^2 - 4}{x^2 + 4}, \ [-4, 4].$$

Solution : Since $x^2 + 4 \geq 4$, the domain for this rational function is all real numbers. So, it is continuous everywhere including the closed interval $[-4, 4]$. Therefore, the function restricted to the interval $[-4, 4]$ achieves both a global max and a global min. The only candidates for the x-coordinate of the min and max are the endpoints of the interval and the critical points for the function that are inside the interval. So, we will compute a derivative and identify all of the critical points. Indeed,

$$f'(x) = \frac{(x^2 + 4)(2x) - (x^2 - 4)(2x)}{(x^2 + 4)^2}$$

$$= \frac{2x^3 + 8x - 2x^3 + 8x}{(x^2 + 4)^2}$$

$$= \frac{16x}{(x^2 + 4)^2}.$$

Critical points occur when $f'(x) = 0$ and when $f'(x)$ is undefined. The only point where $f'(x) = 0$ is $x - 0$, and since $x^2 + 4 \neq 0$ for any x, we know that the derivative is defined everywhere. So, there is only one critical point and to identify the max and min we simply evaluate the function at the endpoints and the critical point.

$$f(-4) = \frac{3}{5}$$

$$f(0) = -1$$

$$f(4) = \frac{3}{5}.$$

The global maximum of the function $f(x)$ on the interval $[-4, 4]$ is $3/5$, and this maximum occurs when $x = -4$ and when $x = 4$. The global minimum of the function $f(x)$ on the interval $[-4, 4]$ is -1, and this minimum occurs when $x = 0$.

5. (20 points)
Show that the equation $x^3 - 15x + 7 = 0$ has exactly one root in the interval $[-2, 2]$.

Solution : Let $f(x) = x^3 - 15x + 7$ and notice that we are trying to show that there is exactly one solution to the equation $f(x) = 0$, when x is restricted to the interval $[-2, 2]$. The function $f(x)$ is a polynomial, so it is both continuous and differentiable everywhere. Since $f(x)$ is continuous on $[-2, 2]$ and since $f(-2) = 29 > 0$ and $f(2) = -15 < 0$, the intermediate value theorem implies that there is at least one solution to the equation $f(x) = 0$ in between -2 and 2.

To see that there is only one solution in between -2 and 2 we compute a derivative $f'(x) = 3x^2 - 15$ and observe that, since $f'(x)$ is continuous, the only place the function $f(x)$ can change from increasing to decreasing (or vice-versa) is when $f'(x) = 0$. But $f'(x) = 0$ only when $x = \pm\sqrt{5}$ and neither of these points live inside $[-2, 2]$. Therefore, on the interval $[-2, 2]$, the function $f(x)$ is either always increasing or always decreasing and so it cannot turn around to pass through the x-axis more than once. Indeed, to do so would require $f'(x) = 0$ for some x in between -2 and 2 by Rolle's theorem, and we know that $f'(x) \neq 0$ for any x in $[-2, 2]$.