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Test 3
Summer 2007
MTH121 Section 01
July 18, 2007

Directions : You have 60 minutes to complete all 5 problems on this exam. There are a possible 100 points to be earned. You may not use your book, notes, or any graphing/programmable calculator. Please be sure to show all pertinent work. *A correct answer with no work will receive very little credit!* If any portion of the exam is unclear please come to me and I will elaborate provided I can do so without giving away the problem.

1. (20 points)
   Let $f(x) = x^3 - 4x$.

   (a) What is the domain of $f(x)$?

   (b) Where are the x and y intercepts?

   (c) Where are the horizontal/vertical asymptotes?

   (d) Where is $f(x)$ increasing/decreasing?

   (e) Identify all critical points as local maxima/minima and determine their value.

   (f) Where is $f(x)$ concave up/concave down and where are the inflection points?

   (g) Sketch the graph of $f(x)$.

   Solution :

   (a) This function is a polynomial, so its domain is all real numbers.

   (b) If $x = 0$ we find that the y-intercept is $(0, 0)$. Setting $y = 0$ we see that $0 = x^3 - 4x = x(x^2 - 4) = x(x+2)(x-2)$ and so the x-intercepts occur at the points $(-2, 0)$ and $(2, 0)$.

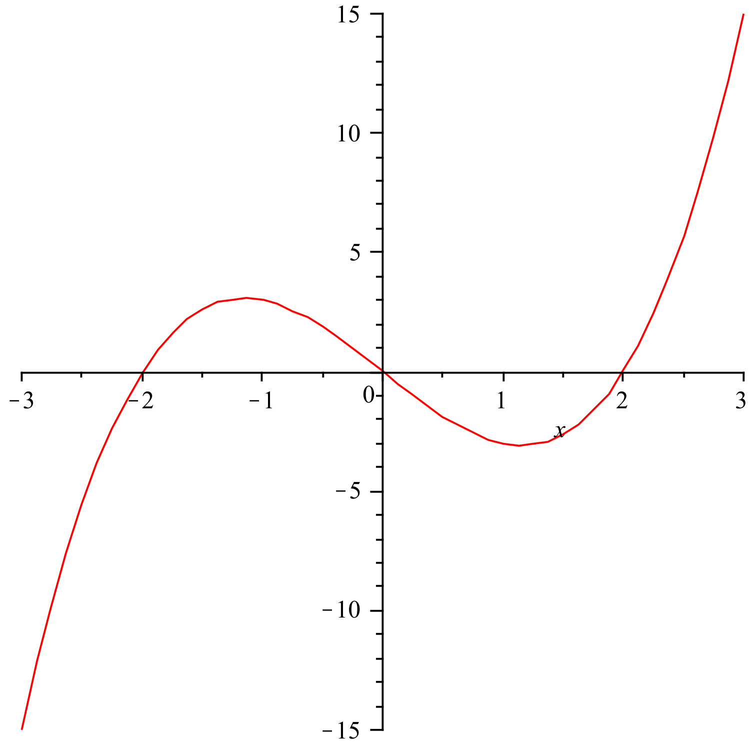
   (c) There are no horizontal or vertical asymptotes.

   (d) Since $f'(x) = 3x^2 - 4 = 0$ when $x = \pm 2/\sqrt{3}$ we have three intervals to check. First we evaluate $f'(x)$ at a point small than $-2/\sqrt{3}$, say at $x = -10$. Indeed, $f - (-10) = 296 > 0$ we know that $f(x)$ is increasing on the interval $(-\infty, -2/\sqrt{3})$. We now choose a value of x in between $-2/\sqrt{3}$ and $2/\sqrt{3}$, say at $x = 0$, and we see that $f'(0) = -4 < 0$. So, the function is decreasing on the interval $(-2/\sqrt{3}, 2\sqrt{3})$. Finally, since $f(10) = 296 > 0$ we know that $f(x)$ is increasing on the interval $(2/\sqrt{3}, \infty)$.

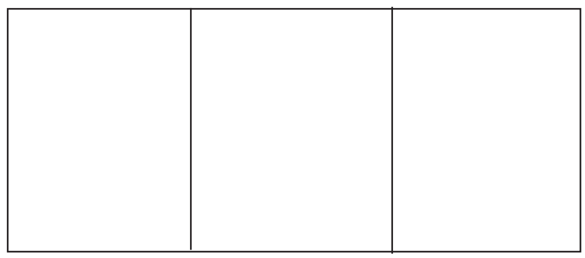
   (e) Since $f(x)$ changes from increasing to decreasing at $x = -2/\sqrt{3}$, there is a local maximum at the point $(-2/\sqrt{3}, 16\sqrt{3}/9)$. Similarly, there is a local minimum at the point $(2/\sqrt{3}, -16\sqrt{3}/9)$.

   (f) To find the intervals of concavity we perform a sign analysis of the second derivative $f''(x) = 6x$ and easily see that $f''(x) > 0$ whenever $x > 0$ and $f''(x) < 0$ whenever $x < 0$. Since the function is defined at the point where the concavity changes, $x = 0$, we know that this is an inflection point.

   (g) The graph is on the following page.


15
10
5
0
−5
−10
−15
−3
−2
−1
1
2
3
x

2. (20 points)
A scientist working for the Umbrella Corporation wishes to enclose a rectangular area into three pens of equal area as seen below. One pen will house those given a fatal dose of the T-virus$^{\text{TM}}$, the second pen will hold those from the first pen that have died and since reanimated as zombies, and the third will hold those who know too much. What are the dimensions that maximize the area provided only 1000 ft of fencing is used to construct the pens?



Solution : With the long lengths of the pen labeled L and the height labeled W we see that the area enclosed is $A = LW$ and the perimeter is $P = 2L+4W$. Since there is only 1000 ft of fencing available we know that $2L+4W = 1000$ and so $L = 500-2W$. Using this in the area equation, we obtain a function of one variable for the area $A(W) = (500-2W)(W) = 500W - 2W^2$. This equation is valid for W in the interval $[0, 250]$ since W is a physical length and so cannot be negative and W must be no more than 250 because if it were larger than this it would force L to be negative. Hence we are trying to maximize a continuous function on a closed interval and so we need only compute the critical points and evaluate the area function at these critical points and the endpoints of the interval $[0, 250]$ to identify the max.

Indeed, $A'(W) = 500 - 4W$ and we see that $A'(W) = 0$ when $W = 125$. To see that this gives the maximum area observe that $A(0) = 0$, $A(125) = 31250$ and $A(250) = 0$. When $W = 125$ we know $L = 250$ (since $L = 500 - 2W$) and this completes the exercise.

3. (20 points)
Use a Riemann sum to approximate the area under the curve $y = x^2 - 2x - 3$ where $2 \leq x \leq 6$ using 4 intervals.

Solution : Since there are only 4 intervals we know that $\Delta x = (6-2)/4 = 1$. Since it was not specified in the problem, I will use the right endpoint approximation.

$$\begin{aligned} R_4 &= f(3)(1)f(4)(1) + f(5)(1) + f(6)(1) \\ &= (3^2 - 2(3) - 3) + (4^2 - 2(4) - 3) + (5^2 - 2(5) - 3) + (6^2 - 2(6) - 3) \\ &= 0 + 5 + 12 + 21 \\ &= 38. \end{aligned}$$

4. (20 points)
Let $v(t) = t^2 - 4t + 3$ represent the velocity of a particle moving on a straight line at time t.

(a) What is the displacement of the particle from time $t = 0$ to time $t = 4$?

(b) What is the total distance the particle traveled from time $t = 0$ to time $t = 4$?

Solution :

(a) The displacement is

$$\int_0^4 v(t)dt = \int_0^4 t^2 - 4t + 3dt$$

$$= \frac{t^3}{3} - \frac{4t^2}{2} + 3t\Big|_0^4$$

$$= \frac{t^3}{3} - 2t^2 + 3t\Big|_0^4$$

$$= \left(\frac{64}{3} - 32 + 3(4)\right) - \left(\frac{0}{3} - 0 + 0\right)$$

$$= \frac{64}{3} - \frac{96}{3} + \frac{36}{3}$$

$$= \frac{4}{3}.$$

(b) The total distance traveled is

$$\int_0^4 |v(t)|dt$$

and we will need to figure out where $v(t)$ is positive and negative. This isn't too bad since $v(t) = t^2 - 4t + 3 = (t - 3)(t - 1)$. Since $v(t)$ is continuous, it must pass through zero to change from positive to negative or from negative to positive and we readily see that $v(t) = 0$ when $t = 1$ and $t = 3$. After checking points around $t = 1$ and $t = 3$, (($v(0) = 3 > 0$,v(2) = -1¡0, and $v(4) = 3 > 0$) we see that $v(t)$ is only negative when t is in between 1 and 3. So,

$$\begin{aligned}
\int_0^4 |v(t)|dt &= \int_0^1 v(t)dt + \int_1^3 -v(t)dt + \int_3^4 v(t)dt \\
&= \int_0^1 v(t)dt - \int_1^3 v(t)dt + \int_3^4 v(t)dt \\
&= \int_0^1 t^2 - 4t + 3dt - \int_1^3 t^2 - 4t + 3dt + \int_3^4 t^2 - 4t + 3dt \\
&= \left(\frac{t^3}{3} - 2t^2 + 3t \Big|_0^1 \right) - \left(\frac{t^3}{3} - 2t^2 + 3t \Big|_1^3 \right) + \left(\frac{t^3}{3} - 2t^2 + 3t \Big|_3^4 \right) \\
&= \left(\frac{4}{3} - 0 \right) - \left(0 - \frac{4}{3} \right) + \left(\frac{4}{3} - 0 \right) \\
&= \frac{12}{3} \\
&= 4.
\end{aligned}$$

5. (20 points)
Compute the following:

a) $\displaystyle \frac{d}{dx}\int_0^{3x^2} 3t\sqrt{t^3 - \sin(t)}dt$ b) $\displaystyle \int_{-\pi/2}^{\pi/2} \cos(\theta)\sin(\sin(\theta))d\theta$

Solution :

(a) This is an exercise in the first version of the Fundamental Theorem of Calculus and the chain rule.

$$\frac{d}{dx}\int_0^{3x^2} 3t\sqrt{t^3 - \sin(t)}dt = \left[3(3x^2)\sqrt{(3x^2)^3 - \sin(3x^2)}\right] 6x$$

(b) Since we see a product of functions, one of which is a composition, we will try to make a u-substitution. Let $u = \sin x$, then $du = \cos(x)dx$ and we have

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} \cos(\theta)\sin(\sin(\theta))d\theta &= \int_{\sin(-\pi/2)}^{\sin(\pi/2)} \sin(u)du \\
&= \int_{-1}^{1} \sin(u)du \\
&= -\cos(u)\Big|_{-1}^{1} \\
&= -\cos(-1) - (-\cos(1)) \\
&= -\cos(-1) + \cos(1) \\
&= -\cos(1) + \cos(1) \text{ (since } \cos(x) \text{ is an even function.)} \\
&= 0.
\end{aligned}$$

An easier solution would ve to observe that the integrand $\cos(\theta)\sin(\sin(\theta))$ is an odd function and we are integrating it across an interval symmetric around 0. So the area exactly cancels.