MATH 321 Section 01
Homework 5

Below is a list of problems that are related to the material we have covered since the last homework set. This homework is optional and is due the day of our final exam Friday May 4. You should write up solutions carefully and neatly and ***staple your work***. All problems from this homework as well as the others are fair game on the final exam! The previous homework and exams will serve as an adequate study guide for the final exam. I will be available during office hours for help.

(1) If A is an $m \times n$ matrix, what is the largest possible value for its rank and the smallest possible value for its nullity?

(2) Let
$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}.$$
Show that A has rank 2 if and only if one or more of the determinants
$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \qquad \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \qquad \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$
is nonzero.

(3) Suppose that A is a 3×3 matrix whose column space is a plane through the origin in 3-space. Can the nullspace be a plane through the origin? Can the row space? Explain.

(4) Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$. Determine which of the following are inner products on $\mathbb{R}^3$. For those that are not, list the axioms that do not hold.
  (a) $\langle \vec{u}, \vec{v} \rangle = u_1v_1 + u_3v_3$
  (b) $\langle \vec{u}, \vec{v} \rangle = u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_3^2$
  (c) $\langle \vec{u}, \vec{v} \rangle = 2u_1v_1 + u_2v_2 + 4u_3v_3$
  (d) $\langle \vec{u}, \vec{v} \rangle = u_1v_1 - u_2v_2 + u_3v_3$

(5) Prove that if $\langle \vec{u}, \vec{v} \rangle$ is the Euclidean inner product on $\mathbb{R}^n$, and if A is an $n \times n$ matrix, then $\langle \vec{u}, A\vec{v} \rangle = \langle A^T\vec{u}, \vec{v} \rangle$. Hint $\langle \vec{u}, \vec{v} \rangle = \vec{u} \cdot \vec{v} = \vec{v}^T\vec{u}$.

(6) Let W be the line in $\mathbb{R}^3$ with equation $y = 2x$. Find an equation for $W^{\perp}$.

(7) Let
$$A = \begin{pmatrix} 1 & 2 & -1 & 2 \\ 3 & 5 & 0 & 4 \\ 1 & 1 & 2 & 0 \end{pmatrix}.$$
  (a) Find bases for the column space of A and nullspace of A^T.

  (b) Verify that every vector in the column space of A is orthogonal to every vector in the nullspace of A^T.

(8) the QR decomposition of a matrix A consists of two matrices Q and R so that $A = QR$ where Q is the matrix whose columns are the result of the Gram-Schmidt process on the columns of A. That is, if A has columns $\vec{c_1}, \vec{c_2}, \ldots, \vec{c_n}$ and if the Gram-Schmidt algorithm used on these columns generates the orthonormal vectors $\vec{q_1}, \vec{q_2}, \ldots, \vec{q_n}$ then Q is the matrix with columns $\vec{q_1}, \vec{q_2}, \ldots, \vec{q_n}$. R is the matrix that you need to multiply with Q to rebuild A. Show that
$$R = \begin{pmatrix} \langle \vec{c}_1, \vec{q}_1 \rangle & \langle \vec{c}_2, \vec{q}_1 \rangle & \cdots & \langle \vec{c}_n, \vec{q}_1 \rangle \\ 0 & \langle \vec{c}_2, \vec{q}_2 \rangle & \cdots & \langle \vec{c}_n, \vec{q}_2 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \langle \vec{c}_n, \vec{q}_n \rangle \end{pmatrix}$$

(9) Compute the QR decomposition for the following matrices and check your answer.
$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

(10) Find a matrix P and a diagonal matrix D so that $A = P^{-1}DP$ where

$$A = \begin{pmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

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