The Algebra Dance

I had initially begun the pursuit of learning how to achieve this paradox as a result of a conversation I had on the subject while I was an undergraduate student. A professor was offering a reading course to me on the basics of measure theory, and, predictably, the concession that there exist non-measurable sets was presented as a necessary evil to the theory. From there, sparked by previous conversations about the Axiom of Choice, the dialog went something like this:

> **Professor :** As a matter of fact it turns out there is a theorem, which requires the Axiom of Choice, which states that a pea can be cut into *finitely* many pieces, rearranged, then glued back together to have a ball the size of the Earth.
>
> **Me :** Wow, how is that possible?
>
> **Professor :** Well the sets used are really bizarre.
>
> **Me :** What do they look like?
>
> **Professor :** That's outside the scope of this course.

His last statement, repeated in several of my analysis books prior to jumping ship on the explanation, would ring in my ears for two years until I finally had a good excuse, and time, to study the paradox. Interestingly enough, while the sets must indeed be bizarre, the proof I will cover reveals little insight to their form. Instead, and possibly more interestingly, the emphasis is placed on *how the pieces are moved around*! The goal of this presentation is to describe exactly how the pieces are moved around.

We start with a definition.

(1) Definition: Let G be a group acting on a set X and suppose $E \subseteq \mathrm{X}$. E is G-paradoxical if for some positive integers m, n there are pair-wise disjoint subsets $A_1, \cdots, A_n, B_1 \cdots B_m$ of E and $g_1, \cdots, g_n, h_1 \cdots h_m \in G$ such that $E = \bigcup_{i=1}^{n} g_i(A_i)$ and $E = \bigcup_{i=1}^{m} h_i(B_i)$.

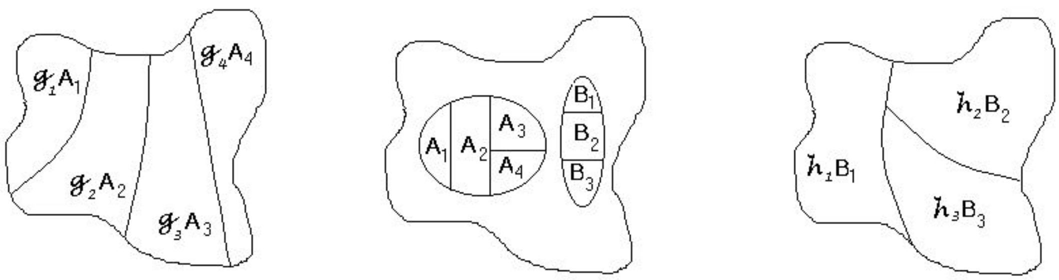

Figure 1

It is interesting to notice that $\left(\bigcup_{i=1}^{n}(A_i)\right) \cup \left(\bigcup_{i=1}^{m}(B_i)\right) \subseteq \mathrm{X}$, not necessarily all of X. The next theorem provides an example to illustrate this concept.

(2) Theorem : A free group F of rank 2 is F-paradoxical, where F acts on itself by left multiplication.

Proof : Suppose that F is generated by α and β. The words in F are derived from the alphabet consisting of two letters, α and β, and each word must therefore begin with one of these letters, with the exception of the empty word denoted by $\{1\}$. Now, define $w(\alpha)$ to be all words in F which start (on the left) with α and define $w(\beta)$, $w(\alpha^{-1})$, and $w(\beta^{-1})$ similarly. Thus we have, using our new notation, another representation for F. That is, $F = \{1\} \cup w(\alpha) \cup w(\alpha^{-1}) \cup w(\beta) \cup w(\beta^{-1})$.

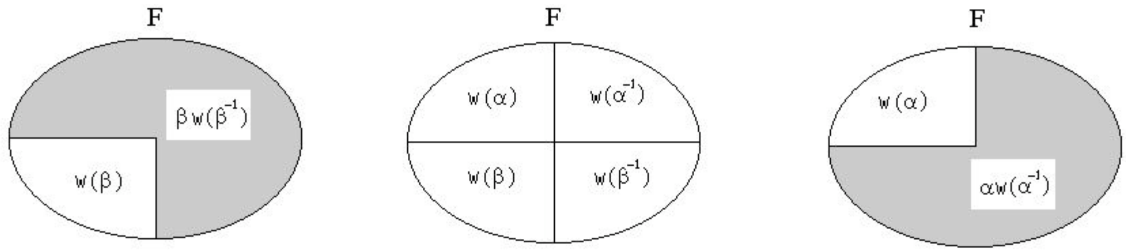

Figure 2

Since $\{1\}(w(\alpha)) \cup \alpha(w(\alpha^{-1})) = F$ and $\{1\}(w(\beta)) \cup \beta(w(\beta^{-1})) = F$, we are done. To verify, suppose $\xi \in F \setminus w(\alpha)$, then $\alpha^{-1}\xi \in w(\alpha^{-1})$ and therefore $\xi = \alpha(\alpha^{-1}\xi) \in \alpha(w(\alpha^{-1}))$ as desired. *QED*

Next we establish a link between the abstract notion of a paradoxical free group and something tied more closely to the spatial world. First we need to recall a definition from algebra.

Definition : Let G be a group and let X be a set. The G-orbit of $x \in \mathrm{X}$ is the set

$$O_x = \{gx \mid g \in G\}.$$

Note that each $x \in \mathrm{X}$ is in some orbit, for if 1 is the identity in G, then $x = 1x \in \{gx \mid g \in G\}$.

Ex. Let $G = \left\{ r_\theta : \theta = \frac{n\pi}{4}, n \in \{0,1,\ldots,7\} \right\}$ where r_θ is counterclockwise rotation through the angle θ. Let $\mathrm{X} = S^1$, then each orbit consists of 8 points. The illustration to the right illustrates the orbit $O_{(0,1)}$. Note that $O_{(1,0)}$, $O_{(-1,0)}$, and $O_{(0,-1)}$ represent the same orbit.

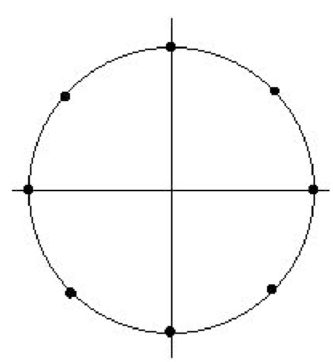

Figure 3

Now, suppose we have a group G that operates on a set X and that G is G-paradoxical. What does the paradoxical nature of G have to do with the paradoxical nature of X? We will show that if G is G-paradoxical then X is G-paradoxical. Since G is paradoxical, we know that there exist pair-wise disjoint subsets $A_1,\dots,A_n,B_1,\dots,B_m$ of G as well as elements of G $g_1,\dots,g_n,h_1,\dots,h_m$ which satisfy the definition. In order to show the set X is paradoxical, we use the Axiom of Choice and orbits to establish a link to the group. By AC there must exist a set, call it Θ which contains exactly one element from each G-orbit in X. The set $\{g(\Theta) \mid g \in G\}$ certainly covers X since Θ contains one $x \in X$ from each orbit left multiplied by each $g_\alpha \in G$, thus regenerating all orbits. If we further assume that G acts on X without any (nontrivial) fixed points, then $\{g(\Theta) \mid g \in G\}$ forms a pair-wise disjoint partition of X. That is $\{g(\Theta) \mid g \in G\}$ would then be a pair-wise disjoint family, which covers X. To see this assume there exist $x, y \in \Theta$ and $g_1, g_2 \in G$ so that $g_1 x = g_2 y \in \{g(\Theta) \mid g \in G\}$. Then notice that this implies that $x = g_1^{-1}(g_2 y)$. Thus, x is another element in the orbit of y which contradicts $x, y \in \Theta$ since we only chose *one* element from each orbit. The previous contradiction will fail if it turns out that $x = y$. However, this implies that $g_1 x = g_2 x$ and so the group element $g_1^{-1} g_2 \in G$ fixes x, contradicting the lack of nontrivial fixed points.

Next, let $A_i^* = \bigcup\{g(\Theta) \mid g \in A_i\}$ and $B_j^* = \bigcup\{g(\Theta) \mid g \in B_j\}$ and notice that since $\{A_i\} \cup \{B_j\}$ are pair-wise disjoint subsets of G, $\{A_i^*\} \cup \{B_j^*\}$ are pair-wise disjoint subsets of X. Finally, for the coup de grâs, recall that since G is paradoxical we have $G = \bigcup_i g_i(A_i) = \bigcup_j h_j(B_j)$ and, therefore, by using the associativity of G and the fact that $\{g(\Theta) \mid g \in G\}$ forms a partition, we can deduce $X = \bigcup_i g_i(A_i^*) = \bigcup_j h_j(B_j^*)$. Indeed, $\bigcup_i g_i(A_i^*) = \bigcup_i g_i\left(\bigcup\{g(\Theta) \mid g \in A_i\}\right)$ which using associativity and the paradoxical nature of G gives us $\{g(\Theta) \mid g \in G\}$ a partition of X. This gives us the following theorem.

Theorem : If G is paradoxical and acts on X without nontrivial fixed points, then X is G - paradoxical.

Thus we have the immediate corollary,

(3) Corollary : X is F - paradoxical whenever F, a free group of rank 2, acts on X with nontrivial fixed points.

To proceed, we identify a free group in $\Re^3$ with two generators. One of the features of the Banach-Tarski paradox is that the way the pieces are moved around is via *isometries*. This limits the generators of a rank 2 free group to two possible candidates, rotations and reflections. It will be shown in the next statement that rotations provide the necessary group structure.

(4) Theorem : SO_3 has a free subgroup of rank 2.

The proof of this statement is provided by Stan Wagon and is given below. The idea behind the proof is to show, inductively, that any non-identity reduced word in the alphabet of rotations around two coordinate axis will not generate the identity matrix. The way this is done is by showing that the general form of any such word acting on a standard basis element can not possibly give back that basis element, thus eliminating the possibility of the word generating the identity matrix.

Proof : Let φ and ψ be counterclockwise rotations around the z-axis and x-axis, respectively, each through the angle $\cos^{-1}\left(\frac{1}{3}\right)$. Then $\varphi^{\pm}$ and $\psi^{\pm}$ are represented by matrices as follows:

$$\varphi^{\pm} = \begin{pmatrix} \frac{1}{3} & \mp\frac{2\sqrt{2}}{3} & \\ \pm\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \psi^{\pm} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \mp\frac{2\sqrt{2}}{3} \\ 0 & \pm\frac{2\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix}$$

We wish to show that no nontrivial word in $\varphi^{\pm}, \psi^{\pm}$ equals the identity. Since conjugation by φ does not affect whether or not a word acts as the identity, we may restrict ourselves to words ending (on the right) in $\varphi^{\pm}$. Hence to get a contradiction, assume that ω is such a word and ω equals the identity.

We claim that $\omega(1,0,0)$ has the form $\left(a, b\sqrt{2}, c\right)/3^{k}$ where a, b, c are integers and b is not divisible by 3. Note, 0 is divisible by any integer. This implies that $\omega(1,0,0) \neq (1,0,0)$ which is the required contradiction. The claim is proved by induction on the length of ω. If ω has length one, then $\omega = \phi^{\pm 1}$ and $\omega(1,0,0) = \left(1, \pm 2\sqrt{2}, 0\right)/3$. Suppose then that $\omega = \phi^{\pm 1}\omega'$ or $\omega = \psi^{\pm 1}\omega'$ where $\omega'(1,0,0) = \left(a', b'\sqrt{2}, c'\right)/3^{k-1}$. A single application of the matrices above shows that $\omega(1,0,0) = \left(a, b\sqrt{2}, c\right)/3^{k}$ where $a = a' \mp 4b'$, $b = b' \pm 2a'$, and $c = 3c'$, or $a = 3a'$, $b = b' \mp 2c'$, and $c = c' \pm 4b'$ according as ω begins with $\phi^{\pm 1}$ or $\psi^{\pm 1}$. It follows that a, b, c are always integers.

It remains only to show that b never becomes divisible by 3. Four cases arise according as ω equals $\varphi^{\pm 1}\psi^{\pm 1}v$, $\psi^{\pm 1}\varphi^{\pm 1}v$, $\varphi^{\pm 1}\varphi^{\pm 1}v$, or $\psi^{\pm 1}\psi^{\pm 1}v$ where, possibly, v is the empty word. In the first two cases, using the notation and equations of the previous paragraph, $b = b' \mp 2c'$ where 3 divides c' or $b = b' \pm 2a'$ where 3 divides a'. Thus if b' is not divisible by 3, neither is b. For the other two cases, let a'', b'', c'' be the integers arising in $v(1,0,0)$. Then in either case, $b = 2b' - 9b''$. For instance, in the third case, $b = b' \pm 2a' = b' \pm 2\left(a'' \mp 4b''\right) = b' + b'' \pm 2a'' - 9b'' = 2b' - 9b''$; an essentially identical proof works in the fourth case. Thus if b' is not divisible by 3, neither is b, completing the proof. $_{QED}$

At this point we would like to combine the results (3) and (4) so that we could produce a paradoxical subset of $\mathfrak{R}^3$ using the group of rotations. Unfortunately, a problem arises when trying to apply these results to a particularly fundamental object, S^2.

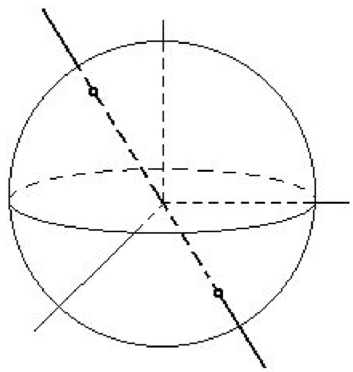

Figure 4

Define the group of rotations from (4) to be $\Im$. Then notice that any element from $\Im$ will fix exactly two elements of S^2 where the axis of rotation intersects the sphere. Thus we may not use (3), yet. Note however that the group $\Im$ is *countable* since it consists of words containing only finitely many syllables, thus so is the set of points of S^2 that it fixes, call it Ω. Then the set $S^2 \setminus \Omega$ still has uncountably many elements and $\Im$ acts on it without nontrivial fixed points. We may want to be careful and be sure that if $p \in S^2 \setminus \Omega$ then $g(p) \in S^2 \setminus \Omega$ as well for all $g \in \Im$. So, proceed by contradiction. That is, assume that $p \in S^2 \setminus \Omega$ and $g(p) \notin S^2 \setminus \Omega$, then $g(p) \in \Omega$. So that means there is an element $f \in \Im$ so that $f[g(p)] = g(p)$. However, if this were true we would have the following, $f[g(p)] = g(p) \Rightarrow (g^{-1}fg)(p) = p$ and so $g^{-1}fg$ fixes p contradicting $p \in S^2 \setminus \Omega$. We have worked out the following result, which is the first real taste of the Banach-Tarski paradox.

(5) Theorem : (Hausdorff Paradox) (AC) There is a countable subset Ω of S^2 such that $S^2 \setminus \Omega$ is SO_3- paradoxical.

We next need to know what we mean when we say the set A *looks like* the set B. The version for "looks like" we will use in this presentation is known as equidecomposable.

(6) Definition : Suppose G acts on X and $A, B \subseteq X$. Then A and B are G-equidecomposable ($A \sim_G B$) if A and B can each be partitioned into the same finite number of respectively G-congruent pieces. Formally,

$$A = \bigcup_{i=1}^{n} A_i , \quad B = \bigcup_{i=1}^{n} B_i$$

$A_i \cap A_j = \emptyset = B_i \cap B_j$ if $i < j \le n$, and there are $g_1, \ldots, g_n \in G$ such that, for each $i \le n$ $g_i(A_i) = B_i$.

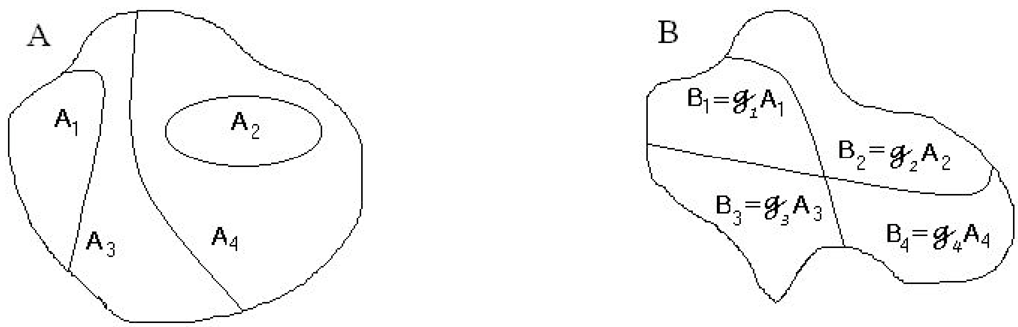

Figure 5

There is another nice property about the above definition.

Proposition : $\sim_G$ defines an equivalence relation.

Proof : Suppose G acts on X and $A, B, C \subseteq X$.

i) Since $\{1\} \in G$ we immediately have $\{1\}A = A$ and so $A \sim_G A$.

ii) Assume that $A \sim_G B$. Then we know there exist $\{A_i\}_{i=1}^{n}$ and $\{B_i\}_{i=1}^{n}$ which are each pair-wise disjoint partitions for A and B, respectively, so that for each $i \le n$ $g_i(A_i) = B_i$ according to the definition of $\sim_G$. Then since G is a group it necessarily contains an inverse for each of its elements and so we also have, $A_i = g_i^{-1}(B_i)$ and thus $B \sim_G A$.

iii) Assume that $A \sim_G B$ and $B \sim_G C$. Then we have pair-wise disjoint families $\{A_i\}_{i=1}^{n}$, $\{B_i\}_{i=1}^{n}$, $\{\hat{B}\}_{j=1}^{m}$, and $\{C_j\}_{j=1}^{m}$ as well as group elements

$g_1,\ldots,g_n,h_1,\ldots h_m \in G$ so that for each $i \le n$, $g_i(A_i) = B_i$ and for each $j \le m$, $h_j(\hat{B}_j) = C_j$. Now further partition A into at most mxn pair-wise disjoint pieces using the following scheme,

$$\begin{aligned}
\hat{A}_1 &= g_1^{-1}(B_1 \cap \hat{B}_1),\\
\hat{A}_2 &= g_1^{-1}(B_2 \cap \hat{B}_1),\\
&\vdots\\
\hat{A}_n &= g_1^{-1}(B_n \cap \hat{B}_1),\\
\hat{A}_{n+1} &= g_2^{-1}(B_1 \cap \hat{B}_2),\\
\hat{A}_{n+2} &= g_2^{-1}(B_2 \cap \hat{B}_2),\\
&\vdots\\
\hat{A}_{nm-1} &= g_n^{-1}(B_{n-1} \cap \hat{B}_m),\\
\hat{A}_{nm} &= g_n^{-1}(B_n \cap \hat{B}_m).
\end{aligned}$$

Next, define the following maps,

$$\begin{aligned}
k_1 &= g_1,\\
k_2 &= g_2,\\
&\vdots\\
k_n &= g_n,\\
k_{n+1} &= g_1,\\
k_{n+2} &= g_2,\\
&\vdots\\
k_{nm-1} &= g_{n-1},\\
k_{nm} &= g_n.
\end{aligned}$$

Finally, notice that for all $i = 1,\ldots,n$, and $\alpha = 0,\ldots,m-1$

$$\begin{aligned}
h_\alpha\left(k_{\alpha n+i}\left(\hat{A}_{\alpha n+i}\right)\right) &= h_\alpha\left(k_{\alpha n+i}\left(g_i^{-1}\left(B_i \cap \hat{B}_\alpha\right)\right)\right)\\
&= h_\alpha\left(g_i\left(g_i^{-1}\left(B_i \cap \hat{B}_\alpha\right)\right)\right)\\
&= h_\alpha\left(\left(B_i \cap \hat{B}_\alpha\right)\right) \subseteq C_\alpha,
\end{aligned}$$

and that $\bigcup_{i=1}^{n} h_\alpha\left(k_{\alpha n+i}\left(\hat{A}_{\alpha n+i}\right)\right) = C_\alpha$. See figure 6 for a visual interpretation.

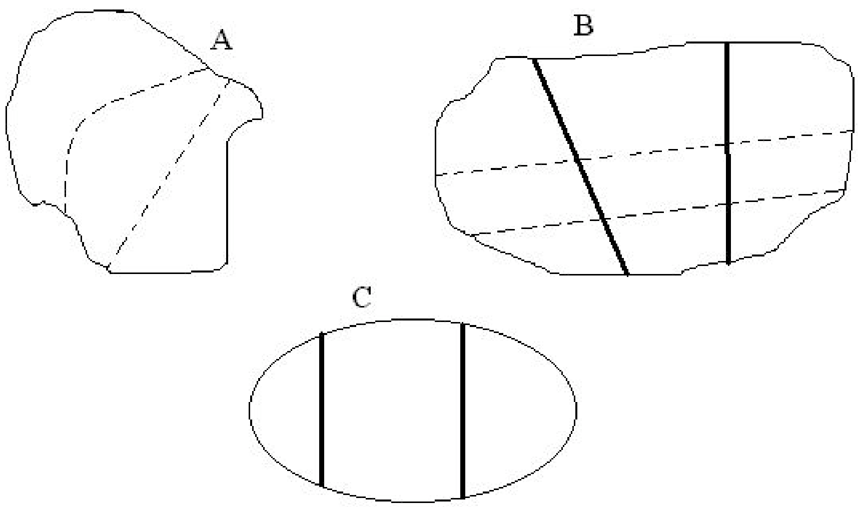

Figure 6

Therefore, it follows that $A \sim_G B$ and $B \sim_G C \Rightarrow A \sim_G C$. Combining i), ii), and iii) we have shown that $\sim_G$ is an equivalence relation.

Using this notion of equidecomposability, and the fact that it defines an equivalence relation, we can now redefine our notion of paradoxical in a more useful way.

Definition : (G - paradoxical II) E is G - paradoxical II if E contains disjoint sets A and B such that $A \sim_G E$ and $B \sim_G E$.

For an illustration, see Figure 1.

Lemma: E is G-paradoxical II iff E is G-paradoxical.

Proof : If E is G - paradoxical then there exist pair-wise disjoint subsets $A_1,\ldots,A_n,B_1,\ldots,B_m$ of E and $g_1,\ldots,g_n,h_1,\ldots,h_m \in G$ such that $E = \bigcup_{i=1}^{n} g_i(A_i)$ and $E = \bigcup_{i=1}^{m} h_i(B_i)$. We are to show there are disjoint sets A and B such that $A \sim_G E$ and $B \sim_G E$.

Let $A = \bigcup_{i=1}^{n} A_i$ and $B = \bigcup_{i=1}^{m} B_i$. Then notice $g_i(A_i) \in E$ so define $E_i = g_i(A_i)$. Is it true that $E_i \cap E_j = \varnothing$ for all $i < j \le n$? If not there are $i \neq j$ so that $E_i \cap E_j \neq \varnothing$ and, hence, elements $x \in A_i$ and $y \in A_j$ so that $g_i(x) = g_j(y)$. However, we need not be redundant, that is we only need either $x \in A_i$ or $y \in A_j$ not both. Remove either x from A_i or y from A_j. Continuing in this manner it is possible to restrict the sets A_i so that $C_i = g_i(A_i)$ is a partition of E. Then, by construction, we have $A \sim_G E$. The case to show $B \sim_G E$ is identical.

Now assume there exist disjoint sets A and B subsets of E such that $A \sim_G E$ and $B \sim_G E$. We are to show that E is G - paradoxical. This is immediate. Since $A \sim_G E$ we know there exist $A_1,\ldots,A_n \in A$ such that $A = \bigcup_{i=1}^{n} A_i$, $A_i \cap A_j = \varnothing$ for all $i < j \le n$, and there exist $g_1,\ldots,g_n \in G$ such that $g_i(A_i) = E_i$, where $\bigcup_{i=1}^{n} E_i = E$ and $E_i \cap E_j = \varnothing$ for all $i < j \le n$. Similarly for $B \sim_G E$. Finally since A and B are disjoint we are done. Therefore our new definition is equivalent to the former. $_{QED}$

If E and E' are equidecomposable then there is a way to break them up into the same number of pieces and have maps which take one piece from E directly onto a piece of E'. So then, if E or E' also happens to be paradoxical then there should be some way to refine the equidecomposable maps so that pieces used to show $E \sim E'$ disjointly cover the pieces which represent the paradoxical nature. Thus, using the obvious composition of maps, we see that both sets would necessarily be paradoxical. The proof of the next theorem is very similar to the proof of the transitivity of the equidecomposable relation $\sim_G$.

(7) Theorem : Suppose G acts on X and E, E' are G - equidecomposable subsets of X. If E is G - paradoxical, so is E'.

The proof of this statement is similar to the proof of the transitivity of the $\sim_G$ equivalence relation and will be omitted.

We now define a partial ordering of the $\sim_G$ classes. First we introduce some new notation.

Notation : Suppose a set A is equidecomposable to a *subset* of another set B with respect to some group G. Then we write, $A \prec B$.

To show that $\prec$ defines a partial order, we must show that $\prec$ is reflexive, antisymmetric, and transitive. Fortunately two of these properties have already been shown. The fact that $\prec$ is reflexive follows from the fact that any set is equidecomposable to itself via the identity map and any partition. The transitivity follows from the previous proof of transitivity for the relation $\sim_G$. The very desirable property of antisymmetry is not quite so obvious and is a result credited to Banach, Schröder, and Bernstein.

(8) Theorem : (Banach – Schröder – Bernstein) Suppose G acts on X and $A, B \subseteq \mathrm{X}$. If $A \prec B$ and $B \prec A$, then $A \sim_G B$. Thus $\prec$ is a partial ordering of the $\sim_G$ classes in $\wp(\mathrm{X})$.

The proof of this theorem hinges on the use of two lemmata.

Lemma 1 : If $A \sim_G B$ then there is a bijection $g : A \to B$ such that $C \sim_G g(C)$ whenever $C \subseteq A$.

Lemma 2 : If $A_1 \cap A_2 = \varnothing = B_1 \cap B_2$, and if $A_1 \sim_G B_1$ and $A_2 \sim_G B_2$, then $(A_1 \cup A_2) \sim_G (B_1 \cup B_2)$.

Proof (Lemma 1) : Using the maps that witness $A \sim_G B$ we can easily define a piece-wise map which takes A onto B. The fact that this map is 1-1 follows from the fact that each element contributing to the piece-wise map is an invertible group element acting on X which contains both A and B. So, formally, if A_i, B_i, and $\{g_i\}_{i=1}^n \in G$ are the sets and maps which witness $A \sim_G B$ then, $g := \begin{cases} g_1, \forall a \in A_1 \\ g_2, \forall a \in A_2 \\ \vdots \\ g_n, \forall a \in A_n \end{cases}$ is the desired bijection. It is then an immediate consequence that $C \sim_G g(C)$ whenever $C \subseteq A$. $_{QED}$

Proof (Lemma 2) : This is almost immediate. Let A_1^i, B_1^i, and $\{g_i\}_{i=1}^n$ be the sets and maps which witness $A_1 \sim_G B_1$ and let A_2^i, B_2^i, and $\{h_i\}_{i=1}^m$ be the sets and maps which witness $A_2 \sim_G B_2$. Then since, $A_1 \cap A_2 = \varnothing = B_1 \cap B_2$ define $C_i = \begin{cases} A_1^i, i = 1, \ldots, n \\ A_2^{i-n}, i = n+1, \ldots, n+m \end{cases}$ and also define $l_i = \begin{cases} g_i, i = 1, \ldots, n \\ h_{i-n}, i = n+1, \ldots, n+m \end{cases}$. Thus we have constructed the sets and maps which witness $(A_1 \cup A_2) \sim_G (B_1 \cup B_2)$. $_{QED}$

Proof (Banach – Schröder – Bernstein) : Since $A \prec B$ we know there exists a set $B_1 \subseteq B$ so that $A \sim_G B_1$ and because $B \prec A$ there exists a set $A_1 \subseteq A$ so that $B \sim_G A_1$. Thus by lemma 1 we know there exist bijections $f : A \to B_1$ and $g : A_1 \to B$ so that if $C_A \subseteq A$ and $C_{A_1} \subseteq A_1$ then $C_A \sim_G f(C_A)$ and $C_{A_1} \sim_G g(C_{A_1})$. Now, define $C_0 = A \setminus A_1$ and inductively define $C_n = g^{-1}f(C_{n-1})$ and let $C = \bigcup_{n=0}^{\infty} C_n$.

Claim : $g(A \setminus C) = B \setminus f(C)$

Pf : It should be noted that the above map $g(A \setminus C)$ *is* indeed acting only on its domain. That is, the set $C_o = A \setminus A_1 \subseteq C$, therefore $(A \setminus C) \subseteq A_1$ and so $g(A \setminus C)$ is well defined. Notice that $C = g^{-1}f(C) \Rightarrow g(C) = f(C)$. Then, since g is the bijection constructed using the maps witnessing $B \sim_G A_1$ we have $g(A \setminus C) = B \setminus f(C)$.

Therefore, we have from lemma 1 $(A \setminus C) \sim_G (B \setminus f(C))$ and $f(C) \sim_G C$. So, using lemma 2 we finally have $C \cup (A \setminus C) \sim_G f(C) \cup (B \setminus f(C)) \Rightarrow A \sim_G B$ as desired. $_{QED}$

We are now in a position to get an improvement on Hausdorff's paradox. Recall that we were able to show that $S^2 \setminus \Omega$ is SO_3 paradoxical. The question now is, how much does $S^2 \setminus \Omega$ look like S^2? Well, if we use the notion of equidecomposability to define "looks like" then the answer is *they look the same*! We will construct two different representations for S^2 and $S^2 \setminus \Omega$, respectively, which are equidecomposable. To do this we show that a certain rotation will, in a sense, absorb the fixed points. Indeed, suppose ρ is a rotation so that $\rho(\Omega), \rho^2(\Omega), \ldots$ is a pair-wise disjoint sequence of sets. Then, if we define $\overline{\Omega} = \bigcup_{n=0}^{\infty} \rho^n(\Omega)$, we have $S^2 = \overline{\Omega} \cup (S^2 \setminus \Omega)$. Next, notice that this isn't much different from $S^2 \setminus \Omega$. Indeed, $S^2 \setminus \Omega = \rho(\overline{\Omega}) \cup (S^2 \setminus \Omega)$ since the only copy of Ω in the set $\overline{\Omega} \cup (S^2 \setminus \Omega)$ is removed when $\overline{\Omega}$ is left multiplied by ρ. Therefore, S^2 is decomposed into two distinct pieces and $S^2 \setminus \Omega$ is decomposed into two distinct pieces with maps (ρ and $\{1\}$) which take one decomposition directly onto the other we have $S^2 \sim S^2 \setminus \Omega$. All that remains is to show that such a rotation exists.

(9) Theorem : If Ω is a countable subset of S^2, then S^2 and $S^2 \setminus \Omega$ are SO_3-equidecomposable.

Proof: Since Ω is a countable subset there must be a line ℓ which passes through the origin and does not intersect Ω. We now show exactly what the rotation *can't* be, then show that there are choices remaining to choose from. Since Ω is countable, give it the representation $\Omega = \{x_n\}_{n=1}^{\infty}$ and for each $x_n \in \Omega$ define Φ_n as the set of all angles θ so that when x_n is rotated about the line ℓ through θ radians it falls back into Ω. Since Ω is countable, Φ_n is countable and therefore $\bigcup_{n=1}^{\infty} \Phi_n$ is also countable. Thus let ρ be any angle so that $\rho \notin \bigcup_{n=1}^{\infty} \Phi_n$. Then we immediately have $\rho^n(\Omega) \cap \Omega = \varnothing$ by construction and so it follows that $\rho^n(\Omega) \cap \rho^m(\Omega) = \varnothing$ whenever $0 \leq m < n$. This follows because $\rho^n(\Omega) \cap \rho^m(\Omega) = \rho^{n-m}(\Omega) \cap \rho(\Omega) = \varnothing$. $_{QED}$

The following is an interesting corollary to this result which, when combined with previous results, yields the strong form of the Banach – Tarski Paradox.

(10) Corollary : (Banach – Tarski Paradox weak form) (AC) S^2 is SO_3- paradoxical, as is any sphere centered at the origin. Moreover, any solid ball in $\Re^3$ is G_3- paradoxical where G_3 is the group of all isometries on $\Re^3$, and $\Re^3$ is itself paradoxical.

Proof: The previous theorem (9) shows that S^2 contains two equidecomposable subsets, S^2 and $S^2 \setminus \Omega$, one of which, $S^2 \setminus \Omega$, is paradoxical according to the Hausdorff Paradox (5). Thus, by theorem (7) we immediately have S^2 is paradoxical. Notice further that none of the arguments use the radius of the sphere in any way. So, then, any sphere centered at the origin in $\Re^3$ is paradoxical.

To verify the statement that any solid ball in $\Re^3$ is G_3- paradoxical first note that since G_3 contains all translations it suffices to assume that we are dealing with a ball centered at the origin. Observe that the unit ball take away the origin deformation retracts onto S^2 via the map $x \mapsto \frac{x}{\|x\|}$. Indeed, the unit ball minus the origin ($B(0,1) \setminus 0$) can use the same decomposition as the sphere. Visualize this as the laying $S^2 \setminus \Omega$ on top of $B(0,1)$ and removing the lines in the ball which connect points in Ω to the origin. Then we have a continuum of shells which are all paradoxical by the same isometries, so we may do it all at once so to speak. So, all we need to show is that the origin can somehow be absorbed. To do this use the same trick involved in the last proof to absorb the countable set Ω. Let $p = \left(0,0,\frac{1}{2}\right)$, and let ρ be a rotation about an axis which passes through p and misses the origin. Then further notice that when the origin is rotated about this axis it will never go outside of the ball. So we have,
$B(0,1) \setminus (0,0,0) \cup \rho\{\rho^n(0,0,0) \mid n = 0,1,2,\ldots\} \sim \{\rho^n(0,0,0) \mid n = 0,1,2,\ldots\} \cup B(0,1)$ similar to the previous proof. Since $\Re^3 \setminus (0,0,0)$ also deformation retracts onto S^2 using the radial contraction map, we also have $\Re^3$ is paradoxical. $_{QED}$

Finally, using the weak form of the paradox we can say something even a little more surprising.

(11) **Theorem :** (Banach – Tarski Paradox strong form) (AC) If A and B are any two bounded subsets of $\Re^3$, each having nonempty interior, then A and B are equidecomposable.

Proof: Since A and B are arbitrary bounded sets we only need to show $A \prec B$ since showing $B \prec A$ would be the same. Then using result (8) we will be done.

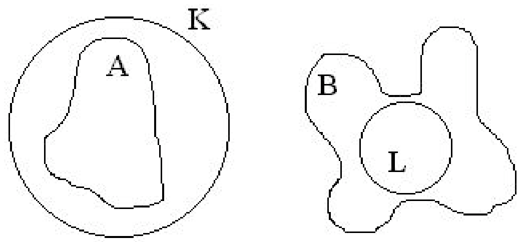

Figure 7

Since A and B are bounded with nonempty interior we can encase A in a solid ball K and let L be a solid ball contained in B. Without loss of generality assume that K has larger volume than L. Since both are bounded, there exists an integer n so that K can be covered by n copies of L.

Now suppose S is a set of n *disjoint* copies of L. Use the result of theorem (7) to construct n paradoxical subsets of L. Next use the weak form of the Banach – Tarski Paradox to generate n copies of L using these subsets and then use translations to move the copies to obtain $S \prec L$.

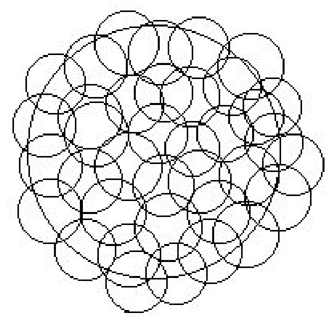

Figure 8

Since we know that we can cover the set K with n copies of L we have $K \prec S$ via the identity map or translations alone. Thus, $A \subseteq K \prec S \prec L \subseteq B$, and so $A \prec B$ and we are done. $_{QED}$

To see that the above usage of the weak form of the Banach – Tarski Paradox along with Theorem (7) will generate a cover of the set S recall the following facts. L is paradoxical and therefore contatins two sets which are each equidecomposable to L, thus by Theorem (7) each of these sets are paradoxical since L is paradoxical. One of these sets can be rotated via the predefined isometries to generate a copy of L and then translated over part of S. Then, since the other set is paradoxical, it contatins two subsets which are equidecomposable to the whole set, and by transitivity they each are equidecomposable to L. They are also each paradoxical by Theorem (7). Take one of these two, generate a copy of L and translate over S and use the other to generate two more paradoxical sets. Repeat this procedure as many times as necessary.

There was only one source used for this paper. All of the above ma terial was an elaboration on the first three chapters of a marvelous book writtten by Stan Wagon. The specifics are as follows:

Wagon, S. (1985). The Banach-Tarski Paradox. Cambridge: Cambridge University Press.