MATH 2200 Fall 2002
Homework 1
Selected Solutions

§2.1 # 34) Write equations for the two straight lines that pass through the point $(2, 5)$ and are tangent to the parabola $y = 4x - x^2$. (*Suggestion:* Draw a figure like Fig 2.1.26.)

Solution : As per the suggestion, first we draw a picture.

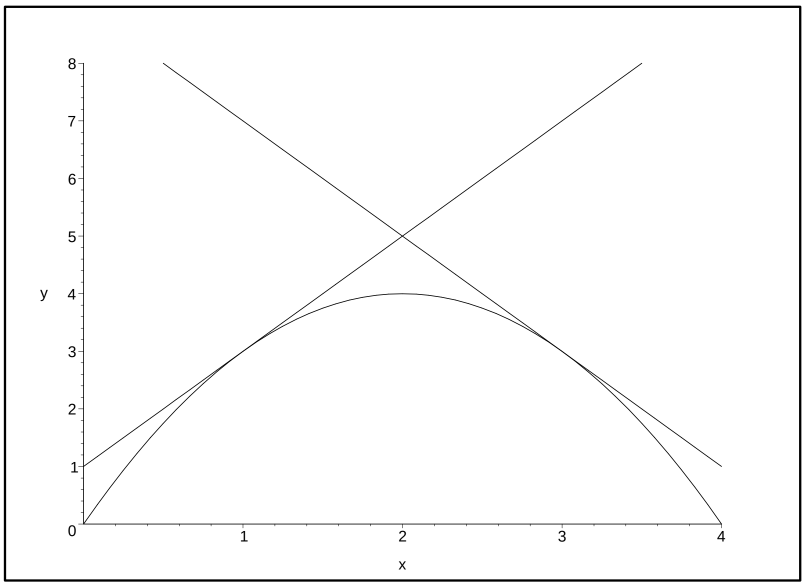

Figure 1: Maple picture of $f(x) = 4x - x^2$

Next we need to write down everything we know. We know that the slope of these lines will have to be $-2a+4$ where the point $(a, 4a-a^2)$ is on the graph of the parabola. This is because we know that the lines are supposed to be tangent to the parabola at a point and we have a formula that will tell us this slope. If the parabola is $y = px^2 + qx + r$, then the slope at the point $(a, pa^2 + qa + r)$ will be $m_a = 2pa+q$. In our case we have $y = 4x - x^2 = (-1)x^2 + 4x + 0$ hence $p = -1$ and $q = 4$ and so $m_a = -2a + 4$. We also know two points on either of the two lines. Each line will pass through the point $(2, 5)$ as well as some point $(a, 4a - a^2)$. So, we can use the point slope form to describe the line.

$$\begin{aligned}
&y - y_1 = m(x - x_1) \\
&\Rightarrow y - 5 = (-2a + 4)(x - 2) \\
&\Rightarrow y = -2ax + 4a + 4x - 8 + 5 \\
&\Rightarrow y = 4x - 2ax + 4a - 3
\end{aligned}$$

Above I used the point $(2, 5)$ along with the slope m_a to describe the lines. We still have one more piece of information. We know that the point $(a, 4a-a^2)$

must also lie on each line. So, let's substitute this in for the x and y.

$$\begin{aligned}
&y = 4x - 2ax + 4a - 3 \\
&\Rightarrow (4a - a^2) = 4a - 2a(a) + 4a - 3 \\
&\Rightarrow a^2 - 4a + 3 = 0 \\
&\Rightarrow (a-1)(a-3) = 0
\end{aligned}$$

Shoot, now we're almost done. We now know the two values of a that make the above equations valid. So, the equation of our lines will be $y = 4x - 2(1)x + 4(1) - 3 = 2x + 1$ and $y = 4x - 2(3)x + 4(3) - 3 = -2x + 9$. (These are the lines that I used in the image above).

§2.2 # 6) Apply the limit laws of this section to evaluate the limit. Justify each step by citing the appropriate limit law.

$$\lim_{t \to -2} \frac{t+2}{t^2+4}$$

Solution :

$$\lim_{t \to -2} \frac{t+2}{t^2+4} = \frac{(\lim_{t \to -2} t + 2)}{(\lim_{t \to -2} t^2 + 4)} \qquad \text{(quotient law)}$$

$$= \frac{(\lim_{t \to -2} t) + (\lim_{t \to -2} 2)}{(\lim_{t \to -2} t^2) + (\lim_{t \to -2} 4)} \qquad \text{(sum law)}$$

$$= \frac{(\lim_{t \to -2} t) + 2}{(\lim_{t \to -2} t^2) + 4} \qquad \text{(constant law)}$$

$$= \frac{-2+2}{(\lim_{t \to -2} t^2) + 4} \qquad \text{(identity law)}$$

$$= \frac{-2+2}{(\lim_{t \to -2} t)\,(\lim_{t \to -2} t) + 4} \qquad \text{(product law)}$$

$$= \frac{-2+2}{(-2)(-2)+4} \qquad \text{(identity law)}$$

$$= \frac{0}{8}$$

$$= 0$$

§2.2 31) Evaluate the limit provided it exists.

Solution :

$$\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2} = \lim_{x\to 4} \frac{(\sqrt{x}+2)(\sqrt{x}-2)}{\sqrt{x}-2}$$

$$= \lim_{x\to 4} \frac{\sqrt{x}+2}{1}$$

$$= 4$$

α) Recall on Monday how we were able to find the slope of a tangent line to the parabola $d(t) = t^2$ at any point $(a, d(a))$, and that this slope was $m_a = 2a$. Use similar techniques to find the slope of the tangent line to the graph of $f(t) = t^3$. Repeat this procedure for $g(t) = t$ and $h(t) = 1$. Do you see a pattern emerging? What would you guess the slope of the tangent line to the graph of $r(x) = x^n$ is at the point $(a, r(a))$ (where n is some nonnegative integer). You may *not* use the word derivative anywhere in your exposition.

Solution : Well to do this problem we have to start with the definition. Recall that the slope m_a of the tangent line to the graph $y = f(x)$ at the point $(a, f(a))$ is given by the following (provided it exists)

$$m_a = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

So, here we go

$$\begin{aligned}
m_a &= \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right) \\
&= \lim_{h \to 0} \left(\frac{(a+h)^3 - a^3}{h} \right) \\
&= \lim_{h \to 0} \left(\frac{(a+h)(a+h)(a+h) - a^3}{h} \right) \\
&= \lim_{h \to 0} \left(\frac{(a^2 + 2ah + h^2)(a+h) - a^3}{h} \right) \\
&= \lim_{h \to 0} \left(\frac{a^3 + 2a^2h + ah^2 + a^2h + 2ah^2 + h^3 - a^3}{h} \right) \\
&= \lim_{h \to 0} \left(\frac{3a^2h + 3ah^2 + h^3}{h} \right) \\
&= \lim_{h \to 0} \left(3a^2 + 3ah + h^2 \right) \\
&= \lim_{h \to 0} \left(3a^2 \right) + \lim_{h \to 0} (3ah) + \lim_{h \to 0} \left(h^2 \right) \\
&= 3a^2 + 0 + 0 \\
&= 3a^2
\end{aligned}$$

Next we move on to an easier computation. We simply need to evaluate the limit of the difference function for $g(t) = t$ and $h(t) = 1$. Drum roll please,

$$
\begin{aligned}
m_a &= \lim_{h\to 0} \frac{g(a+h)-g(a)}{h} \\
&= \lim_{h\to 0} \left(\frac{(a+h)-a}{h}\right) \\
&= \lim_{h\to 0} \left(\frac{h}{h}\right) \\
&= \lim_{h\to 0} 1 \\
&= 1
\end{aligned}
$$

Interesting, it's *always* 1. Well okay, maybe not too interesting. After the graph of $g(t) = t$ is a line and lines have constant slope. Finally, we need to do the same computation for the constant function $h(t) = 1$. Bring it on,

$$
\begin{aligned}
m_a &= \lim_{h\to 0} \frac{h(a+h)-h(a)}{h} \\
&= \lim_{h\to 0} \left(\frac{1-1}{h}\right) \qquad \text{(remember } h(t) \text{ is identically 1, so } h(a+h)=1\text{)} \\
&= \lim_{h\to 0} \left(\frac{0}{h}\right) \\
&= \lim_{h\to 0} 0 \\
&= 0
\end{aligned}
$$

Is that hard to believe? Nope, the graph of $h(t) = 1$ is just a horizontal line. So, the rise over run is always zero.

Now, as far as pattern recognition goes we have the following sequence. $m_a = 0$ for $y = 1$, $m_a = 1$ for $y = t$, $m_a = 2a$ for $y = t^2$, and $m_a = 3t^2$ for $y = t^3$. You can work some more examples if you want more data to base a hypothesis on but it is the case that $m_a = nx^{n-1}$ for $y = x^n$. Bing badda boom were done.

β) Now that we know that the limit of a sum is the sum of the limits, provided they exist, (or at least we will know after Wednesday or Friday) use your results from problem α) to find the slope of the tangent line at the point $(a, p(a))$ of the graph of the polynomial $p(t) = 1 + t + t^2 + \cdots + t^n$.

Solution : Now that we have our guess we only need to use the sum law and product law to find the desired slope. Indeed,

$$m_a = \lim_{h \to 0} \frac{p(a+h) - p(a)}{h}$$

$$= \lim_{h \to 0} \frac{(1 + (a+h) + (a+h)^2 + \cdots (a+h)^n) - (1 + a + a^2 + \cdots + a^n)}{h}$$

$$= \lim_{h \to 0} \left(\frac{1-1}{h} + \frac{(a+h) - a}{h} + \frac{(a+h)^2 - a^2}{h} + \cdots + \frac{(a+h)^n - a^n}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{1-1}{h} \right) + \lim_{h \to 0} \left(\frac{(a+h) - a}{h} \right) + \lim_{h \to 0} \left(\frac{(a+h)^2 - a}{h} \right) + \cdots \lim_{h \to 0} \left(\frac{(a+h)^n - a^n}{h} \right)$$

(by the sum law)

$$= 0 + 1 + 2t + \cdots + nx^{n-1}$$

(by our guess in α)