MATH 2200 Fall 2002
Homework 2
Selected Solutions

§2.2 # 41) Use the four-step process illustrated in Examples 12 and 13 to find a slope-predictor function for the given function $f(x)$. Then write an equation for the line tangent to the curve $y = f(x)$ at the point where $x = 2$.

$$f(x) = \frac{2}{x-1}$$

Solution : The four steps that are mentioned are the following,

1. Write the definition of $m(x)$.
2. Substitute into this definition the formula for the given function f.
3. Make algebraic simplifications until Step 4 can be carried out.
4. Determine the value of the limit as $h \to 0$.

So, let us begin. First we need to write down the definition. This is easy enough,

$$m(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

.

Now we need to substitute our function f into the above equation. This gives us,

$$m(x) = \lim_{h \to 0} \frac{\frac{2}{(x+h)-1} - \frac{2}{x-1}}{h}.$$

Now notice that we would like to be able to just use the quotient law to compute the limit. But, as is *always* the case with the limit of the difference quotient, we cannot do this immediately because the denominator approaches zero. Thus we need to slam into step 3 and perform some algebraic manipulations. Indeed,

$$
\begin{aligned}
m(x) &= \lim_{h\to 0} \frac{\frac{2}{(x+h)-1} - \frac{2}{x-1}}{h} \\
&= \lim_{h\to 0} \frac{\frac{2(x-1)-2(x+h-1)}{(x+h-1)(x-1)}}{h} \\
&= \lim_{h\to 0} \frac{2x-2-2x-2h+2}{(x+h-1)(x-1)h} \\
&= \lim_{h\to 0} \frac{-2h}{(x+h-1)(x-1)h} \\
&= \lim_{h\to 0} \frac{-2}{(x+h-1)(x-1)} \qquad \text{(Now we can use the quotient law!)} \\
&= \frac{-2}{(x-1)(x-1)} \\
&= \frac{-2}{(x-1)^2}
\end{aligned}
$$

Now that we have computed the slope-predictor function (otherwise known as the derivative) we can find the equation of the desired line. We know from the directions that the line must pass through the point $(2, f(2))$ and be tangent to the graph $y = f(x)$ at that point. Well, if it is to be tangent to the graph when $x = 2$ then we know that the slope of the line must be $m(2)$, after all, that is what the slope-predictor function tells you. So, we know the slope and we know a point on the line. That's all we need. The desired line will be the following,

$$y - f(2) = m(2)(x-2).$$

Since $f(2) = 2/(2-1) = 2$ and $m(2) = -2/(2-1)^2 = 2$ the above reduces to,

$$
\begin{aligned}
y - 2 &= -2(x-2) \\
\Rightarrow y &= -2x + 6.
\end{aligned}
$$

§2.3 # 3) Evaluate the following limit

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta^2}$$

Solution : We cannot immediately use the quotient law because the denominator limits to zero. However, the numerator also limits to zero so there is still some hope. Since we can't factor anything let's try multiplying by 1 in disguise using the conjugate trick.

$$\begin{aligned}
\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta^2} &= \lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta^2} \left(\frac{1 + \cos\theta}{1 + \cos\theta} \right) \\
&= \lim_{\theta \to 0} \frac{1 - \cos^2\theta}{\theta^2(1 + \cos\theta)} \\
&= \lim_{\theta \to 0} \frac{\sin^2\theta}{\theta^2(1 + \cos\theta)} \\
&= \lim_{\theta \to 0} \left(\frac{\sin\theta}{\theta} \right) \left(\frac{\sin\theta}{\theta} \right) \left(\frac{1}{1 + \cos\theta} \right) \\
&= \left(\lim_{\theta \to 0} \frac{\sin\theta}{\theta} \right) \left(\lim_{\theta \to 0} \frac{\sin\theta}{\theta} \right) \left(\lim_{\theta \to 0} \frac{1}{1 + \cos\theta} \right) \\
&= (1)(1)(\frac{1}{2}) \\
&= \frac{1}{2}
\end{aligned}$$

§2.3 # 19) Evaluate the following limit

$$\lim_{z\to 0} \frac{\tan z}{\sin 2z}$$

Solution : Since one of the only limit facts we have from this section is that $\lim_{\theta\to 0} \sin\theta/\theta = 1$ it is likely that we will need to use this at some point. Notice the following fact that will come in handy for this problem,

$$\lim_{\theta\to 0} \frac{\theta}{\sin\theta} = \lim_{\theta\to 0} \frac{1}{\left(\frac{\sin\theta}{\theta}\right)} = \frac{1}{\lim_{\theta\to 0} \frac{\sin\theta}{\theta}} = \frac{1}{1} = 1.$$

Here we go,

$$\begin{aligned}
\lim_{z\to 0} \frac{\tan z}{\sin 2z} &= \lim_{z\to 0} \frac{\sin z}{\cos z \sin 2z} \\
&= \lim_{z\to 0} \frac{\sin z}{\cos z \sin 2z} \left(\frac{2z}{2z}\right) \\
&= \lim_{z\to 0} \left(\frac{2z}{\sin 2z}\right) \left(\frac{\sin z}{2z\cos z}\right) \\
&= \lim_{z\to 0} \left(\frac{2z}{\sin 2z}\right) \left(\frac{\sin z}{z}\right) \left(\frac{1}{2\cos z}\right) \\
&= \left(\lim_{z\to 0} \frac{2z}{\sin 2z}\right) \left(\lim_{z\to 0} \frac{\sin z}{z}\right) \left(\lim_{z\to 0} \frac{1}{2\cos z}\right) \\
&= (1)\,(1) \left(\frac{1}{2}\right) \\
&= \frac{1}{2}
\end{aligned}$$

§2.3 # 51) There is exactly one point a where both the right-hand and left-hand limits of $f(x)$ fail to exist. Describe as in Example 10) the behavior of $f(x)$ near a.

$$f(x) = \frac{x-1}{x+1}$$

Solution : It should be fairly clear that something interesting will happen near the point $x = -1$. After all, this is the point at which the denominator becomes zero. Since the numerator near this point is close to -2 we see that there is indeed some type of asymptotic behavior near $x = -1$ (that is f blows up at the point $x = -1$ and we need to investigate how it blows up). For all values of x close to -1 we see that the numerator will be negative. Indeed, the numerator doesn't change sign until $x = 1$ which is relatively far away from our point of interest. So, we know that the denominator approaches zero as $x \to -1$ and we only need to do a little sign analysis to determine the type of blow up. If we let x approach -1 from the left then we know that the values in the denominator will be approaching zero from the negative side. If x is approaching -1 from the left then we know that $x + 1 < 0$. Similarly, if we approach -1 from the right then we know the denominator will always be positive. Therefore, since the numerator is always negative in a small neighborhood of $x = -1$, we see that we have the following,

$$\lim_{x\to -1^-} \frac{x-1}{x+1} \longrightarrow +\infty \qquad \text{and} \quad \lim_{x\to -1^+} \frac{x-1}{x+1} \longrightarrow -\infty$$

One can see this from the graph below.

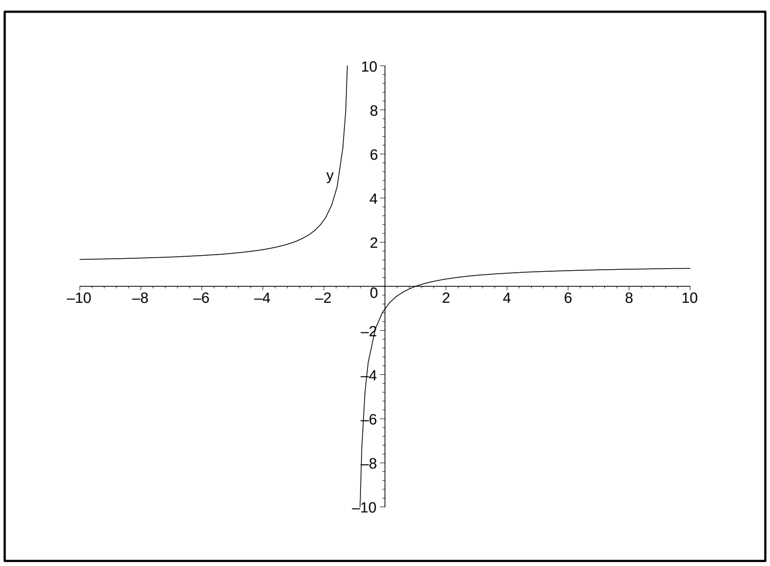

Figure 1: $f(x) = \frac{x-1}{x+1}$

α) Have a good break.

Solution : My solution was to go to Amsterdam and do some math. Solutions can, and probably will, vary.