MATH 2200 Fall 2002
Homework 3
Selected Solutions

§ 2.4 # 3) Apply the limit laws and the theorems of this section to show that the given function is continuous for all x.

$$g(x) = \frac{2x-1}{4x^2+1}$$

Solution : We will need to show that for any real number a, the function $g(x)$ is continuous at a. That is, we will need to show that $g(a)$ is defined for each a and that

$$\lim_{x \to a} g(x) = g(a).$$

Since $g(x)$ is a fraction of polynomials we know that it will be defined for all values of x so that the denominator is not zero. Notice that the denominator is the polynomial $4x^2 + 1$. If x is any real number, then x^2 is either positive or zero. So, $4x^2$ is either positive or zero and thus $4x^2+1$ must be strictly positive. Since zero is not a positive number we deduce that $4x^2 + 1 \neq 0$, and so $g(x)$ is defined everywhere.

It remains to show

$$\lim_{x \to a} g(x) = g(a).$$

But this is immediate by the quotient law. The quotient law says that

$$\lim_{x \to a} \frac{f(x)}{h(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} h(x)}$$

provided that the limit in the denominator is non-zero. Since we showed previously that the denominator is never zero (in fact it is always at least 1) we know that we can use the quotient law. That is, we can evaluate the desired limit by computing the limit of the numerator and denominator and divide them. Well, the numerator and denominator of $g(x)$ are *polynomials*, so we know that they are continuous. So, to evaluate the limit we can just plug in the value. That is,

$$\lim_{x \to a} g(x) = \lim_{x \to a} \frac{2x-1}{4x^2+1} = \frac{\lim_{x \to a}(2x-1)}{\lim_{x \to a} 4x^2+1} = \frac{2a-1}{4a^2+1} = g(a).$$

So, the function g is continuous at every real number.

§ 2.4 # 15) Tell where the following function is continuous.

$$f(x) = 2x + x^{\frac{1}{3}}$$

Solution : This guy is continuous for all real numbers. Recall that we can use the power law to push a limit inside a root provided it is either an odd root or if it is an even root we need to be sure that the limit of the term under the radical is nonnegative. In this case we need to compute the limit of a cube root. Since 3 is odd there is no restriction on using the power law. Notice we can also always use the sum law and the product law (provided the sum of the limits and the product of the limits exist). Thus we have,

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\longrightarrow a} 2x + x^{\frac{1}{3}} \\
&= \lim_{x\to a} 2x + \lim_{x\to a} x^{\frac{1}{3}} && \text{(by the sum law)} \\
&= 2a + \left(\lim_{x\to a} x\right)^{\frac{1}{3}} && \text{(by the power law (among others))} \\
&= 2a + a^{\frac{1}{3}} \\
&= f(a).
\end{aligned}$$

So, since $f(x)$ is defined for all x and since the limit of $f(x)$ as x approaches a tends to $f(a)$ we see that we have satisfied the definition of continuity. Thus, $f(x)$ is continuous for all real numbers.

§ 2.4 # 33) Tell where the following function is continuous.

$$f(x) = \frac{1}{\sin 2x}$$

Solution : Since $\sin 2x$ is continuous everywhere, we know that the function $f(x)$ will be continuous provided the denominator is non-zero. Indeed, if $\sin 2a \neq 0$, then by the quotient law we have,

$$\lim_{x \to a} f(x) = \lim_{x \to a} \frac{1}{\sin 2x} = \frac{\lim_{x \to a} 1}{\lim_{x \to a} \sin 2x} = \frac{1}{\sin 2a} = f(a).$$

So, we know that the function will be continuous where $\sin 2x \neq 0$, so it remains to find all such x. We know that $\sin x = 0$ when $x = n\pi$ where n is any integer. So, $\sin 2x = 0$ when $2x = n\pi$. Thus when $x = \frac{n\pi}{2}$, $f(x)$ will be undefined, hence discontinuous. These are the only points so that $f(x)$ is discontinuous (as our computation above shows).

Chapter 2 Miscellaneous Problems # 61) Apply the intermediate value property of continuous functions to prove that the equation $x^5 + x = 1$ has a solution.

Solution : Let $f(x) = x^5 + x - 1$. Then $f(x)$ is continuous since it is a polynomial. So, if we can find points a and b so that $f(a) < 0$ and $f(b) > 0$, then by the intermediate value theorem we will know that there is a point c in between a and b so that $f(c) = 0$ ($c^5 + c - 1 = 0 \Rightarrow c^5 + c = 1$ and we will have solved the problem). Note that $f(0) = -1$ and $f(1) = 1$. Thus there exists a solution in between 0 and 1. Sure enough, Maple says that a solution is approximately .7548776662.

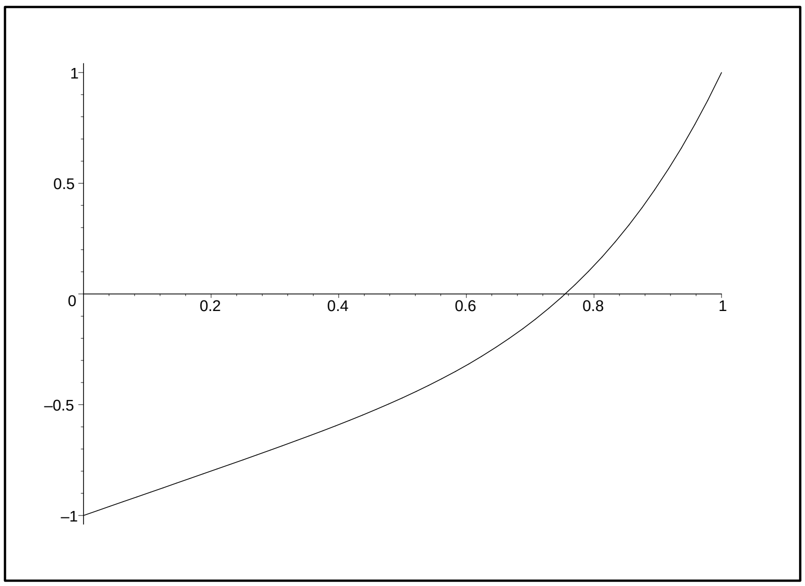

Figure 1: Maple picture of $f(x) = x^5 + x - 1$