MATH 2200 Fall 2002
Homework 6
Selected Solutions

§ 3.4 # 39) Compute the derivative of the following function with respect to x.

$$f(x) = \frac{(2x+1)^{\frac{1}{2}}}{(3x+4)^{\frac{1}{3}}}$$

Solution : We're gonna need the quotient rule *and* the chain rule. My cup runneth over...

$$f'(x) = \frac{\frac{1}{2}(2x+1)^{-\frac{1}{2}}(2)(3x+4)^{\frac{1}{3}} - (2x+1)^{\frac{1}{2}}(\frac{1}{3})(3x+4)^{-\frac{2}{3}}(3)}{(3x+4)^{\frac{2}{3}}}$$

$$= \frac{(2x+1)^{-\frac{1}{2}}(3x+4)^{\frac{1}{3}} - (2x+1)^{\frac{1}{2}}(3x+4)^{-\frac{2}{3}}}{(3x+4)^{\frac{2}{3}}}$$

$$= \frac{(3x+4)^{\frac{1}{3}}}{(2x+1)^{\frac{1}{2}}(3x+4)^{\frac{2}{3}}} - \frac{(2x+1)^{\frac{1}{2}}}{(3x+4)^{\frac{4}{3}}}$$

$$= \frac{3x+4}{(2x+1)^{\frac{1}{2}}(3x+4)^{\frac{4}{3}}} - \frac{2x+1}{(2x+1)^{\frac{1}{2}}(3x+4)^{\frac{4}{3}}}$$

$$= \frac{x+3}{(2x+1)^{\frac{1}{2}}(3x+4)^{\frac{4}{3}}}$$

§ 3.5 # 20) Find the maximum and the minimum values attained by the function $f(x)$ on the interval $[1,3]$, where

$$f(x) = x^2 + \frac{16}{x}.$$

Solution : All we need to use the theorey we developed is a continuous function on a closed interval. As if by magic, the function $f(x)$ is only discontinuous at $x = 0$. Since $0 \notin [1,3]$ (read 0 is not an element of the closed set $[1,3]$) we know that $f(x)$ is continuous on the closed set $[1,3]$. Therefore, we know that $f(x)$ achieves a local maximum and a local minimum inside the closed interval $[1,3]$. (Note that this is not the same as saying that $f(x)$ achieves its *global* maximum and minimum values inside [1,3]! This is certainly not true since $f(x)$ is unbounded, i.e., it gets as big (or small) as I want near $x = 0$.) Moreover, we know that $f(x)$ achieves this local minimum and local maximium either at an endpoint or at a critical point. So, we need to find the critical points. Recall, a critical point is an x value so that $f'(x)$ is either undefined or $f'(x) = 0$. So, lets compute the derivative.

$$f'(x) = 2x - \frac{16}{x^2} = \frac{2x^3 - 16}{x^2}.$$

So, let's find the critical points. First, note that the only point for which $f'(x)$ is undefined is where the denominator is zero. That is when $x = 0$. But this is not a point in our interval $[1,3]$, so we can ignore it. Thus, the only critical points that we care about are when $f'(x) = 0$. Lets find those guys.

$$\begin{aligned} &f'(x) = 0 \\ \Rightarrow &\frac{2x^3 - 16}{x^2} = 0 \\ \Rightarrow &2x^3 - 16 = 0 \\ \Rightarrow &2x^3 = 16 \\ \Rightarrow &x^3 = 8 \\ \Rightarrow &x = 2 \end{aligned}$$

Wow. As if by magic (again) we see that the critical point $x = 2$ *is* in the interval $[1,3]$, so it is a potential local minimum or local maximum. It remains only to evaluate the function at the endpoints and critical points to determine which is largest and which is smallest.

$$\begin{aligned} f(1) &= 1^2 + \frac{16}{1} = 17 \\ f(2) &= 2^2 + \frac{16}{2} = 12 \\ f(3) &= 3^2 + \frac{16}{3} = \frac{43}{3} = 14 + \frac{1}{3} \end{aligned}$$

Thus, we see that $f(x)$ has a local maximum at $x = 1$ and its value is 17, and it has a local minimum at $x = 2$ and its value is 12.

§ 3.5 # 20) Find the maximum and the minimum values attained by the function $f(x)$ on the interval $[0, 3]$, where

$$f(x) = \frac{x}{x^2+1}.$$

Solution : My my my. What a wonderful world it is that we live in. The fates have decreed *again* that the function we are given is continuous on the given interval. Whoa hoss! I need to verify this a little. Why is it true that $f(x)$ is continuous on $[0, 3]$? Well, $f(x)$ is a rational function of polynomials (i.e., it looks like a fraction of polynomials). By the quotient law for limits (ah sweet memories of past ideas) and the fact that polynomials are continuous, we know that $f(x)$ will be continuous at all points so that the denominator is non-zero. So, you gotta ask yourself, "will $x^2 + 1$ ever be zero?" The answer is no, and the reason is that $x^2 + 1 > 0$ since $x^2 \geq 0$ and adding 1 to a non-negative number always yields a positive number. We have shown that $f(x)$ is continuous everywhere, so it is certainly continuous on the closed interval $[0, 3]$. So, we know that it achieves it's local extrema inside this interval at either endpoints or critical points. Let's find the critical points.

$$\begin{aligned} f'(x) &= \frac{1(x^2+1) - x(2x)}{(x^2+1)^2} \\ &= \frac{1-x^2}{(x^2+1)^2} \end{aligned}$$

Notice that the denominator is still going to be strictly positive since it is the square of a function that we already know is strictly positive. Therefore, we know that no critical points will come from points that make the derivative undefined. It remains to find points that make the derivative zero.

$$f'(x) = 0 \Rightarrow \frac{1-x^2}{(x^2+1)^2} = 0 \Rightarrow 1 - x^2 = 0 \Rightarrow x = \pm 1$$

We know that we can throw out the critical point -1 since it is not in the interval of interest. So, we only need to evaluate $f(x)$ at the endpoints and the critical point to determine the local extrema.

$$\begin{aligned} f(0) &= 0 \\ f(1) &= \frac{1}{2} \\ f(3) &= \frac{3}{10} \end{aligned}$$

So, the local maximum occurs when $x = 1$ and its value is $f(1) = 1/2$. The local minimum occurs when $x = 0$ and its value is $f(0) = 0$.

§ 3.5 # 50) Match the given graph of the function with the graph of its derivative f'. (See page 149 in Edwards and Penny).

Solution :

47) C

48) F

49) D

 # 50) B

51) A

52) E