MATH 2200 Fall 2002
Homework 7
Selected Solutions

§ 3.6 Problems 5, 9, 11, 13, 20, 21, 25, 27, 29, 31, 33, 45, 47

α) Let $S^1 = \{(x,y) : x^2 + y^2 = 1\}$ (read "the set of points (x,y) such that $x^2 + y^2 = 1$) be the unit circle centered at the origin. Find, using techniques from §3.6, the closest point in S^1 to the point $(0,1)$.

Solution : Ok ok. So I *meant* to ask you to find the point on S^1 that is closest to the point $(0,2)$. We can still use techniques from this section to solve the case I asked for too.

First we identify what it is that we want to minimize or maximize. Since I want to find the point in S^1 that is closest to the point $(0,1)$ we see that we would like to minimize the distance between points. That is, we would like to minimize the function,

$$d((0,1),(x,y)) = \sqrt{(x-0)^2 + (y-1)^2}.$$

To do this, we had first write this as a function of one variable and then find a closed set that we are interested in. Notice that we are only interested in using points on the circle. Tha is, we only want to compute the distance to points (x,y) that satisfy the equation $x^2 + y^2 = 1$. Well shoot, if I try to solve this equation for y we obtain $y = \pm\sqrt{1-x^2}$, where the positive square root corresponds to the top half of the circle and the negative square root corresponds to the bottom half of the circle. It seems clear that the point $(0,1)$ will be closer to a point in the top half so lets just use the equation $y = +\sqrt{1-x^2}$. Then we see that we want to minimize the equation

$$\begin{aligned} d(x) &= \sqrt{(x-0)^2 + (\sqrt{1-x^2} - 1)^2} \\ &= \sqrt{x^2 + (1-x^2) - 2\sqrt{1-x^2} + 1} \\ &= \sqrt{2 - 2\sqrt{1-x^2}}. \end{aligned}$$

Now, before we can use any of the theory we need to find a closed interval to restrict our analysis to. We know that we only want to use points on the unit circle in the distance equation and we have shown that we only need to know the x-coordinate (since $y = \sqrt{1-x^2}$). So, what are the valid x-coordinates that we can use? Well, all of the x-coordinates of points on the unit circle live in betwen -1 and 1. So, it stands to reason that the closed interval that we are looking for is exactly $[-1,1]$.

We now need to show that $d(x)$ is continuous on the interval $[-1,1]$. Heuristically this is clear. $d(x)$ is the distance between a point on the unit circle with x-coordinate equal to x and the point $(0,1)$. This is always defined for points on the unit circle since there is a well defined distance between points on the unit circle and the point $(0,1)$. But lets see if we can check it analytically. We know that the square root function is continuous on its domain, so we need only to show that the the distance function (which is really just a square root) is defined on the interval $[-1,1]$. That is, we need to show that $(2-2\sqrt{1-x^2}) \geq 0$ for all $x \in [-1,1]$. Then it will follow that $d(x)$ is continuous on $[-1,1]$. So, first lets consider $\sqrt{1-x^2}$. Notice that for all $x \in [-1,1]$ it is true that $1-x^2 \geq 0$. Therefore, we know that $\sqrt{1-x^2}$ is well defined on the closed interval. It is also true that $0 \leq \sqrt{1-x^2} \leq 1$ when $x \in [-1,1]$. Therefore we see that $0 \leq 2\sqrt{1-x^2} \leq 2$, and so $2-2\sqrt{1-x^2} \geq 0$. Thus, we can take its square root and obtain the function $d(x)$, which we have just shown to be continuous on $[-1,1]$.

Great! Now we know that all of the extreme values will occur either at an endpoint of $[-1,1]$ or at critical points of $d(x)$ that live inside $[-1,1]$. Let us now find the critical points. To do this we need to compute the derivative of $d(x)$. To do this, lets rewrite $d(x)$ in a form that may be a little easier to compute with.

$$d(x) = \sqrt{2-2\sqrt{1-x^2}} = \left(2-2\left(1-x^2\right)^{1/2}\right)^{1/2}$$

Now we use the chain rule a couple of times to compute the derivative.

$$\begin{aligned} d'(x) &= \frac{1}{2}\left(2-2\left(1-x^2\right)^{1/2}\right)^{-1/2}\left(-\frac{1}{2}\left(1-x^2\right)^{-1/2}\right)2x \\ &= \frac{x}{-2\left(1-x^2\right)^{1/2}\left(2-2\left(1-x^2\right)^{1/2}\right)^{1/2}} \end{aligned}$$

Now to find the critical points. (Sheesh) Lets first find out where this guy is undefined. The numerator is fine, so we need to check when the denominator is zero. But we already know most of the answer 'cause we already did it. We need to know when the denominator is zero or undefined. But, check it out, one of the big nasty terms in the denominator is exactly $d(x)$! We know that it is well defined for all $x \in [-1,1]$ and is only zero when $x=0$. So we have a critical point when $x=0$. Now, what about the other term in the denominator (the $-2\left(1-x^2\right)^{1/2}$ part). We looked at something very similar to this when we were trying to show $d(x)$ was continuous. We know for all $x \in [-1,1]$ this guy is well defined, and it is only zero when $x = \pm 1$, so we get additional critical points $x=-1$ and $x=1$ to check. For the last bit of critical points we need to check when $d'(x)=0$. Well, this only happens when the numerator is zero, hence when $x=0$ and since we already have this point in our bag of points to check we don't get anything new. So, now we need to check the endpoints $x=-1$ and $x=1$ as well as the critical points $x=-1,0,1$ (again we see some redundancy, that is the endpoints are critical points which is not always the case). Indeed,

$$
\begin{aligned}
f(-1) &= \sqrt{2} \\
f(0) &= 0 \\
f(1) &= \sqrt{2}
\end{aligned}
$$

Hey, we are done. We see that the distance is minimized when $x = 0$. The only point on the top half of the unit circle that has x-coordinate equal to zero is the point $(0, 1)$. So, $(0, 1)$ is the point on the unit circle that is closest to $(0, 1)$. (Note: $(0, 1)$ is *also* the closest point on the unit circle to the point $(0, 2)$ which is a little more interesting).

β) Write a detailed description (in your own words!) of the method you use to solve problems like those in §3.6, citing theorems if you like. You may use a previously worked problem as an example, but you need to carefully explain what you are doing at each step and why. The goal of this exercise is to solidify a plan of attack for word problems. I'll go ahead and tell you now, this problem will be graded. So, ask me in advance if you have questions about this. I'll be more than happy to look at drafts and give you advice. This problem will be worth 30 points out of 50.

Solution : Check your notes. I tend to write out exactly what I am doing and why I am doing it when I present a problem.