

MATH 2200 Fall 2002
Homework 9
Selected Solutions

§3.9 # 25) Use implicit differentiation to find an equation of the line tangent to the curve $x^{2/3} + y^{2/3} = 5$ at the point $(8, 1)$.

Solution : We only need to find the slope of the tangent line.

$$\begin{aligned}\frac{d}{dx} (x^{2/3} + y^{2/3}) &= \frac{d}{dx} (5) \\ \Rightarrow \frac{2}{3}x^{-1/3}\frac{dx}{dx} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} &= 0 \\ \Rightarrow \frac{2}{3}x^{-1/3} &= -\frac{2}{3}y^{-1/3}\frac{dy}{dx} \\ \Rightarrow -\frac{x^{-1/3}}{y^{-1/3}} &= \frac{dy}{dx}\end{aligned}$$

So, the slope at the point $(8, 1)$ is

$$\frac{dy}{dx}(8, 1) = -\frac{8^{-1/3}}{1^{-1/3}} = -\frac{(\frac{1}{2})}{1} = -\frac{1}{2}$$

So, the equation of the line is

$$\begin{aligned}y - 1 &= -\frac{1}{2}(x - 8) \\ \Rightarrow y &= -\frac{1}{2}x + 5\end{aligned}$$

§3.9 # 36) Water is being collected from a block of ice with a square base (see **Figure 3.9.15** on page 196). The water is produced because the ice is melting in such a way that each edge of the base of the block is decreasing at 2in./h while the height of the block is decreasing at 3 in./h. What is the rate of flow of water into the collecting pan when the base has edge length 20 in. and the height of the block is 15 in.? Make the simplifying assumption that the water and the ice have the same density.

Solution : Thank goodness for the simplifying assumption! If it wasn't there we would have to deal with the fact that if we lose 1 unit of volume of ice, we *don't* get 1 unit of volume of water. ick. Fortunately the assumption tells us that the volume exchange is one for one. The volume of this block of ice is given by $V = x^2y$, and we are interested in finding dV/dt at the instant when $x = 20$ and $y = 15$. We also know that, at this instant, $dx/dt = -2$, and $dh/dt = -3$. So, lets compute a derivative and see if we are missing anything.

$$\begin{aligned}\frac{d}{dt}(V) &= \frac{d}{dt}(x^2y) \\ \Rightarrow \frac{dV}{dt} &= 2xy \frac{dx}{dt} + x^2 \frac{dy}{dt}\end{aligned}$$

It looks like we have everything we need. So, substituting in the values we know we see that

$$\frac{dV}{dt} = 2(20)(15)(-2) + (20)^2(-3) = -2400\text{in}^3/\text{h}$$

§3.9 # 39) A circular oil slick of uniform thickness is caused by a spill of 1 m³ of oil. The thickness of the oil slick is decreasing at the rate of 0.1 cm/h. At what rate is the radius of the slick increasing when the radius is 8 m.

Solution : The key idea to remember with this problem is that the volume of the oil slick remains constant. The fact that the oil slick is circular with a uniform thickness tells us that it is a cylinder. We know that $V = 1$ and if h denotes the thickness of the cylinder we also know that $dh/dt = -0.1\text{cm/h} = -0.001\text{m/h}$ (be careful about units!). Finally, we see that we are asked to find dr/dt at the instant when $r = 8$. So, lets compute a derivative of the volume equation and see what else we need to know.

$$\begin{aligned}\frac{d}{dt}(V) &= \frac{d}{dt}(\pi r^2 h) \\ \Rightarrow \frac{dV}{dt} &= 2\pi r \frac{dr}{dt} h + \pi r^2 \frac{dh}{dt}\end{aligned}$$

We know that we are interested in this equation when $r = 8$, and at that time we know $dh/dt = -0.001$. So, it seems that we don't know dV/dt and h . Now, notice that the volume of the oil slick is not changing. The oil slick is expanding, but there is no oil lost. Therefore the rate of change of volume with respect to time is zero (indeed, it ain't changin' at all), that is $dV/dt = 0$. To find h we go back to the volume equation $V = \pi r^2 h$. We know that $V = 1$ and we are interested in computing h when $r = 8$. So, we see that $1 = \pi(8)^2 h \Rightarrow h = 1/(64\pi)$. Now we have everything we need.

$$\begin{aligned}\frac{dV}{dt} &= 2\pi r \frac{dr}{dt} h + \pi r^2 \frac{dh}{dt} \\ \Rightarrow 0 &= 2\pi(8) \frac{dr}{dt} \left(\frac{1}{64\pi}\right) + \pi(8^2)(-0.001) \\ \Rightarrow 0 &= \left(\frac{1}{4}\right) \frac{dr}{dt} - 0.064\pi \\ \Rightarrow 0.256 &= \frac{dr}{dt} \\ \Rightarrow \frac{32\pi}{125} &= \frac{dr}{dt}.\end{aligned}$$

§3.9 # 68) A highway patrol officer's radar unit is parked behind a billboard 200 ft from a long straight stretch of U.S. 17. Down the highway, 200 ft from the point on the highway closest to the officer, is an emergency call box. The officer points the radar gun at the call box. A minivan passes the call box and, at that moment, the radar unit indicates that the *distance between the officer and the minivan* is increasing at 45 mi/h – that is, 66 ft/s. The posted speed limit is 55 mi/h. Does the officer have any reason to apprehend the driver of the minivan?

Solution : We need a picture to help us out with this one.

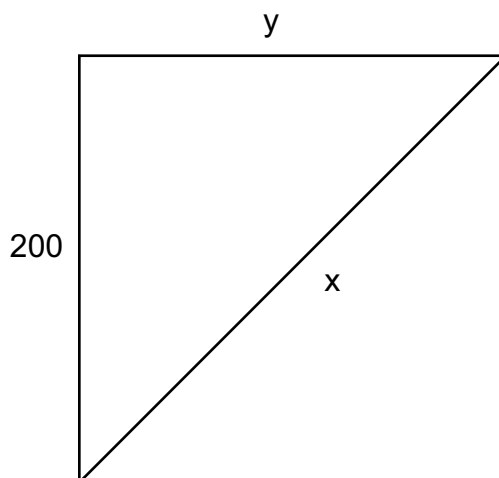


Figure 1: Here we see that the minivan's position on the road is given by the top of the triangle labeled with the variable y . The police officer is located at the vertex formed by the edge of length 200 and the edge of variable length x .

Now we are in business. We have a right triangle and we know that we can use the Pythagorean theorem to relate the variables. Indeed, we have the equation $x^2 = (200)^2 + y^2$ and we are interested in finding the velocity dy/dt at the instant when $y = 200$ and we also know, at this instant, that $dx/dt = 66$. So, let's compute a derivative and see what else, if anything, we need.

$$\frac{d}{dt}(x^2) = \frac{d}{dt}((200)^2 + y^2)$$

$$\Rightarrow 2x \frac{dx}{dt} = 2y \frac{dy}{dt}$$

Looks like we need one more piece of information. We don't yet know what

the variable x is when $y = 200$. But we can figure it out rather easily from the equation $x^2 = (200)^2 + y^2$. Indeed, this equation tells us

$$x^2 = (200)^2 + (200)^2 \Rightarrow x^2 = 80000 \Rightarrow x = 400\sqrt{2}$$

So, now we can stick this new information into the previous equation to obtain the desired result.

$$2x \frac{dx}{dt} = 2y \frac{dy}{dt}$$

$$\Rightarrow 2(400\sqrt{2})(66) = 2(200) \frac{dy}{dt}$$

$$\Rightarrow 66\sqrt{2} = \frac{dy}{dt}$$

$$\Rightarrow 93.33809509 \approx \frac{dy}{dt}$$

Keep in mind that this measurement is in ft/s. So, we need to convert to mi/h to determine if the minivan gets a ticket. Since there are 3600 seconds in an hour and 5280 feet in a mile we see that the minivan is traveling at a speed

$$dy/dt = 93.33809509 \left(\frac{3600}{5280} \right) \approx 63.63961029 \text{mi/h}$$

The driver of the minivan *should* get a ticket.

§3.9 # 57)(As requested by Mr. Babcock) A 10-ft ladder is leaning against a wall. The bottom of the ladder begins to slide away from the wall at a speed of 1 mi/h. (a) Find the rate at which the top of the ladder is moving when it is 4 ft from the ground. If the top of the ladder maintained contact with the wall, find the speed with which it would be moving when it is (b) 1 in. above the ground; (c) 1 mm above the ground. Do you believe your answers? The key to the apparent paradox is that when the top of the ladder is about 1.65 ft high, it disengages altogether from the wall and thereafter slides away from it.

Solution :

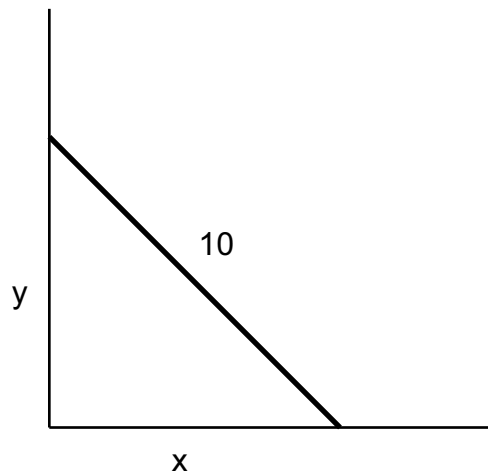


Figure 2: The ladder is represented here by the bold line of length 10. The variable x denotes the distance from the wall to the base of the ladder while the variable y denotes the distance from the floor to the top of the ladder.

From the diagram we see that we are (again) dealing with the equation of Pythagoras. Indeed, we have $100 = x^2 + y^2$. So, since the problem asks for us to compute dy/dt lets compute an implicit derivative and solve for this term.

$$\frac{d}{dt}(100) = \frac{d}{dt}(x^2 + y^2)$$

$$\Rightarrow 0 = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$\Rightarrow \left(-\frac{x}{y}\right) \frac{dx}{dt} = \frac{dy}{dt}$$

So, to solve part (a) we need first to compute everything in similar units.

Because I am familiar with velocity in terms of mi/h I will try to write all of my values in these units. Thus we know that $dx/dt = 1$ and we want to find dy/dt when $y = 4 \text{ ft} = 4/5280 \text{ mi} = 1/1320 \text{ mi}$. So, it looks like we will need to solve for x first. But we know $100 = x^2 + y^2$ (in feet) and if $y = 4$ that tells us that $x = 2\sqrt{21}$ ft which gives us $x = \sqrt{21}/2640 \text{ mi}$ after a conversion. So,

$$\frac{dy}{dt} = \left(-\frac{\left(\frac{\sqrt{21}}{2640}\right)}{\left(\frac{1}{1320}\right)} \right) (1) \approx -2.291287848 \text{ mi/h.}$$

Now, just because I'm sick of the conversions, I'm going to use a little reasoning to deduce the oddity shown by parts (b) and (c). Notice that since $dx/dt = 1$ the equation for the velocity is

$$\frac{dy}{-dt} = -\frac{x}{y}.$$

Now, what happens as the ladder gets closer to the floor? The y values approach zero. Furthermore, notice that the x values will never get any larger than 10 ft, which occurs when the ladder is laying flat on the ground. So, as the height of the ladder approaches zero the velocity becomes becomes arbitrary large (it approaches infinity). This is the paradox mention by parts (b) and (c).