A couple of Max Min Examples

§ 3.6 # 2) Find the maximum possible area of a rectangle of perimeter 200m.

Solution : As is the case with all of these problems, we need to find a function to minimize or maximize and a closed interval on which the function is continuous. The goal is to find a closed interval that defines the region we care about.

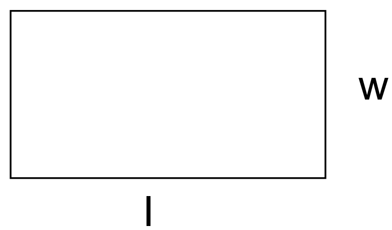

Figure 1: Rectangle of perimeter 200m

We are asked to maximize the area of a rectalngle. So, we have an equation for that $A = lw$. We also know that the perimeter of the rectangle is to be 200m. That is we know $2l + 2w = 200$. So, we need to write the area function as a function of one variable. Fortunately, we know from the constraint on the perimeter that $2l + 2w = 200$ and so $l = 100 - w$. Hence, we can write the area function as a function of w. Indeed, we have $A(w) = (100 - w)w = 100w - w^2$. Now, we will want to find a closed interval so that we can use the theory. We know that the smallest the width can be is zero, thus the left endpoint of the closed interval will be 0. Then we also know that the length can only get as small as zero. So, when the length is zero we see from the perimieter constraint equation that $w = 100$. So, we now have our interval $[0, 100]$ and we want to maximize the area function when $w \in [0, 100]$. The theory tells us that this maximum value must occur either at an endpoint of the interval or at a critical point of the function. So, we will need to check all of these. Let's find the critical points. Recall, a critical point of the function $A(w)$ is a point so that $A'(w) = 0$ or $A'(w)$ is undefined. Since $A'(w) = 100 - 2w$ we see that this will be defined everywhere in our interval $[0, 100]$ (indeed it is defined everywhere). So, the only critical points we can hope to get occur when $A'(w) = 0$. So, $A'(w) = 0$ implies that $100 - 2w = 0$ and this occurs when $w = 50$. Fortunately this value is in our closed interval so we need to check it. Thus to conclude the problem we need to check the value of the area function at each of the endpoints $w = 0$ and $w = 100$, as well as at the critical point $w = 50$.

$$\begin{aligned} A(0) &= 0 \\ A(50) &= 100(50) - 50^2 = 5000 - 2500 = 2500 \\ A(100) &= 0 \end{aligned}$$

For my next trick I'll see if I answered the problem that was asked. We were asked to find the maximum possible area. Did we do that? You betcha', and that maximum value is 2500m^2.

§ 3.6 # 4) A farmer has 600m of fencing with which to enclose a rectangular pen adjacent to a long existing wall. He will use the wall for one side of the pen and the available fencing for the remaining three sides. What is the maximum area that can be enclosed in this way?

Solution : Ok, I'm going to be a little more brief with the rest of these. But the plan of attack is always the same. Find a function to minimize or maximize. Define a closed interval of interest which should be evident by either a given constraint or physical constraints. Use the theory to find all possible points in the interval that can lead to the maximum or minimum value of the function in the interval. Evaluate the function at each one of the critical points and endpoints to see which give the largest and smallest values of the function on the interval.

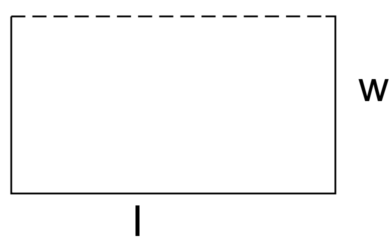

Figure 2: Rectangle with three sides to cover (the dashed line is the side against the wall)

Let's get to work. We want to maximize area $A = lw$. We know that we have 600m of fencing and that we only need to use it on three sides of the enclosure (since the other side is taken up by the wall. Thus we have the constraint $600 = 2w + l$. Thus, we see that $l = 600 - 2w$ and we can write the area function as a function of one variable $A(w) = (600 - 2w)w = 600w - 2w^2$. Next, we need to find the closed interval. One endpoint will be if we have zero width and the other endpoint will be found by supposing that the length is zero. Indeed, if the length is zero, then we see from the constraint equation that $600 = 2w + 0$ and hence $w = 300$. So, the closed interval will be $[0, 300]$.

On to the theory. We know that the only candidates for a maximum in the interval $[0, 300]$ is either an endpoint or a critical point. So, to find the critical points, we take the derivative. $A'(w) = 600 - 4w$, which is defined everywhere, so it is defined in the interval $[0, 300]$. We also get critical points when $A'(w) = 0$ and this occurs only when $w = 150$, which is in the interval. Now it remains to check to see if the critical point or the endpoint will give us the maximum value.

$$
\begin{aligned}
A(0) &= 0 \\
A(150) &= 600(150) - 2(150)^2 = 45000 \\
A(300) &= 0
\end{aligned}
$$

So, the maximum area occurs when the width is 150m and the length is 300m, giving us a maximum area of 45000m^2.

§ 3.6 # 6) If x is in the interval $[0, 1]$, then $x - x^2$ is not negative. What is the maximum value that $x - x^2$ can have on that interval? In other words, what is the greatest amount by which a real number can exceed its square?

Solution :

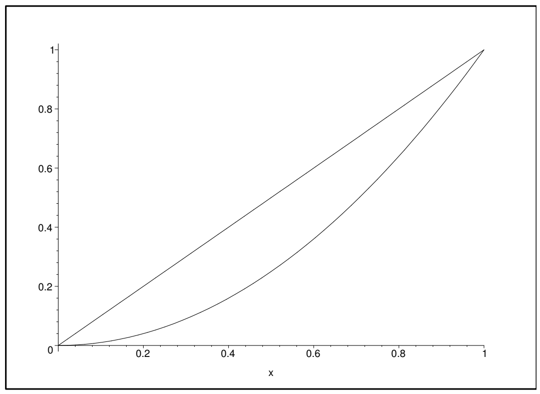

Figure 3: Here we see the graph of the functions $f(x) = x$ and $g(x) = x^2$. Note that only on the closed interval $[0, 1]$ is it true that $x \geq x^2$. The function we are minimizing $(x - x^2)$is the distance between these two functions.

We are already given the function *and* the interval on which to maximize the function. So there's no setup required. We know that the maximum on this interval will occur at either (you guessed it) an endpoint or at a critical point. Let's find the critical points. The derivative of the function will be the function $x \mapsto 1 - 2x$ (whoa! new notation! They didn't give the function a name, so we can say that the function we have is $x \mapsto x - x^2$ which is read "x maps to $x - x^2$", thus its derivative is the function x maps t0 1 - 2x). Since the derivative is defined everywhere we see that the only critical points we will get occur when it is equal to zero. $1 - 2x = 0$ implies $x = 1/2$ which is in the given interval. So, we need to check the possibilities,

$$0 \mapsto 0 - 0^2 = 0$$
$$\frac{1}{2} \mapsto \frac{1}{2} - \frac{1}{2}^2 = \frac{1}{4}$$
$$1 \mapsto 1 - 1^2 = 0$$

That's kinda neat. The greatest amount a real number can exceed its square is exactly $\frac{1}{4}$. Cool.

§ 3.6 # 8) A rectangle of fixed perimeter 36 is rotated about one of its sides, thus sweeping out a figure in the shape of a right circular cylinder (Fig. 3.6.17). What is the maximum possible volume of that cylinder?

Solution :

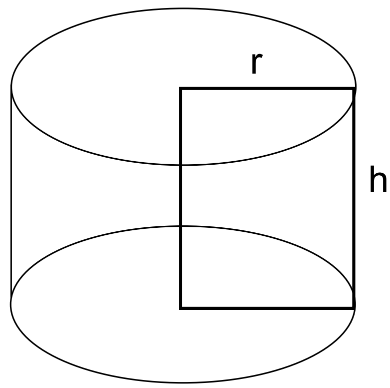

Figure 4: A not so good copy of Fig. 3.6.17 with suggestive variable labelling.

We want to maximize the volume $V = \pi r^2 h$. Let's be a little sneaky and write the perimeter of the rectangle as $2h+2r = 36$. Then we see that $h = 18-r$ and so the volume equation can be expressed as a function of one variable $V(r) = \pi r^2(18 - r) = 18\pi r^2 - \pi r^3$. Moreover, the physical constraints of the problem tell us that we are only concerned with values of r in the closed interval $[0, 18]$ (obtained by the fact that r cannot be negative and finding what r is if $h = 0$ in the perimeter constraint). So, now we find critical points. $V'(r) = 36\pi r - 3\pi r^2 = 3\pi r(12 - r)$. This is a polynomial so it is defined everywhere. The other critical points can occur when $V'(r) = 0$ which happens either when $3\pi r = 0$ or when $12 - r = 0$. Thus $V'(r) = 0$ when $r = 0$ or when $r = 12$. So, we only need to check the endpoints 0 and 18 as well as the critical points 0 (which is redundant) and 12.

$$\begin{aligned} V(0) &= 0 \\ V(12) &= 864\pi \\ V(18) &= 0 \end{aligned}$$

So, the maximum possible volume of the cylinder is 864π cubic whatevers (the units are not given).

§ 3.6 # 10) Suppose that the strength of a rectangular beam is proportional to the product of the width and the *square* of the height of its cross section. What shape beam should be cut from a cylindrical log of radius r to achieve the greatest possible strength?

Solution : First we need to discuss what "is proportional to" means. To say that x is proportional to y (written $x \propto y$) means that there exists some constant c so that $x = cy$. So in the case of this problem we know that the strength (S) is proportional to the product of the width and the *square* of the height of its cross section (wh^2). So, we get the relation $S \propto wh^2$ which means that there is some constant c out there in la-la land so that $S = cwh^2$. I shall assume by "cross section" they mean the circular cross section of the log.

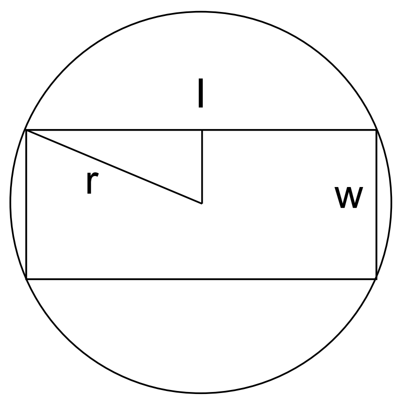

Figure 5: A circular cross section of the log.

We see that we have the equation $r^2 = w^2/4 + h^2/4$ from the triangle in the cross section. Keep in mind that we know r and c are both *constants*. So, if we wanted to write the strength equation as a function of one variable we need to find either h in terms of w or w in terms of h. It may be better to write h in terms of w because otherwise we will have to do some nasty algebra when we substitute into the strength equation. Indeed, $h = \pm\sqrt{4r^2 - w^2}$, and we will take $h = \sqrt{4r^2 - w^2}$ to avoid negative height. So, the strength can now be written as

$$S(w) = cwh^2 = cw(\sqrt{4r^2 - w^2})^2 = cw(4r^2 - w^2) = 4cr^2w - cw^3.$$

Now we need to define a closed interval. Certainly we know that the width must be non-negative so we get the usual left endpoint zero. The right endpoint is found by assuming that the other variable dimension is zero. Indeed, if $h = 0$ then we see from the equation $r^2 = w^2/4 + 0^2$ that $w = \pm 2r$ (again we only care about when $w = +2r$). So, the right endpoint is $2r$. The closed interval is $[0, 2r]$. Next we find the critical points.

$S'(w) = 4cr^2 - 3cw^2$ (remember that c and r are *constants*).

$S'(w)$ is defined everywhere and $S'(w) = 0$ when $w = 2r/\sqrt{3}$, which is in the interval. Let's find which gives us the largest strength.

$$\begin{aligned}
&S(0) = 4cr^2(0) - c(0)^3 = 0 \\
&S\left(\frac{2r}{\sqrt{3}}\right) = 4cr^2\left(\frac{2r}{\sqrt{3}}\right) - c\left(\frac{2r}{\sqrt{3}}\right)^3 = \frac{16cr^3}{3\sqrt{3}} \\
&S(2r) = 8cr^3 - 8cr^3 = 0
\end{aligned}$$

So, the strength is the greatest when the width of the beam is $2r/\sqrt{3}$ and the height is $\sqrt{4r^2 - 4r^2/3}$ giving us a strength of $16cr^3/(3\sqrt{3})$.

§ 3.6 # 12) Find the maximum possible volume of a right circular cylinder if its total surface area-including both circular ends-is 150π.

Solution : The volume is given by the equation $V = \pi r^2 h$ and the surface area constraint is given by $2\pi r^2 + 2\pi r h = 150\pi$.

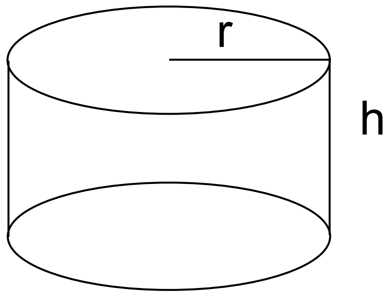

Figure 6: A cylinder.

We can solve the surface area constraint equation for h and obtain

$$h = \frac{150\pi - 2\pi r^2}{2\pi r} = \frac{75 - r^2}{r}.$$

This allows us to rewrite the volume equation as

$$V(r) = \pi r^2 \frac{75 - r^2}{r} = 75\pi r - \pi r^3$$

where $r \in [0, 5\sqrt{3}]$. How did I get the closed interval you may ask? Well, certainly r must be non-negative so we have $r \geq 0$ giving us the left endpoint zero. Then, if the other dimension, h, were zero we see from the surface area constraint equation that $2\pi r^2 + 2\pi r(0) = 150\pi$, which tells us that $r = \sqrt{75} = 5\sqrt{3}$. Now we need to find critical points.

$$V'(r) = 75\pi - 3\pi r^2$$

So, there are no critical points given to us when $V'(r)$ is undefined (since it is defined everywhere). Thus, the only critical points we can get are when $V'(r) = 0$. This happens when $75\pi - 3\pi r^2 = 0$ and this implies that $r = \pm 5$. But -5 is not in the closed interval we have defined, so we throw it away. Thus we need to check the value of the volume at the three points $r = 0, r = 5$, and $r = 5\sqrt{3}$.

$$\begin{aligned}
V(0) &= 75\pi(0) - \pi(0)^3 = 0 \\
V(5) &= 75\pi(5) - \pi(5)^3 = 375\pi - 125\pi = 250\pi \\
V(5\sqrt{3}) &= 75\pi(5\sqrt{3}) - \pi(5\sqrt{3}^3 = 0
\end{aligned}$$

So, we see that the maximum volume will occur when $r = 5$ and $h = 10$, giving us a volume of 250π cubic whatevers (again, no units were given).

§ 3.6 # 14) A rectangle has a line of fixed length L reaching from one vertex to the midpoint of one of its far sides (Fig. 3.6.19). What is the maximum possible area of such a rectangle?

Solution :

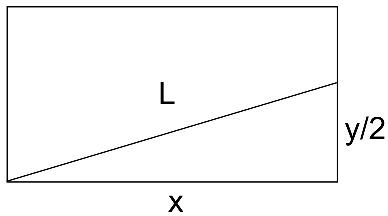

Figure 7: A bad copy of Fig. 3.6.19.

We want to maximize the area of the rectangle $A = xy$. To find the constraint function good 'ol Pythagoras tells us that $L^2 = x^2 + (y/2)^2$. So, we can solve this constraint equation for y to obtain $y = \pm 2\sqrt{L^2 - x^2}$, again we are only concerned with the positive square root. So, we can write the area as a function of one variable $A(x) = 2x\sqrt{L^2 - x^2}$. The variable x must be non-negative and so the left endpoint of the necessary closed interval is again zero. Then, since y must also be non-negative we see that the largest x can be is when $L^2 = x^2 + (0/2)^2$ which tells us that the right endpoint is L. So, the closed interval we have is $[0, L]$. To find the critical points we take the derivative of the area function and look to see where it is zero or undefined in $[0, L]$.

$$
\begin{aligned}
A'(x) &= 2\sqrt{L^2 - x^2} + 2x\left(\frac{1}{2}\right)\left(L^2 - x^2\right)^{-1/2}(-2x) \\
&= \frac{2\left(L^2 - x^2\right) - 2x^2}{\left(L^2 - x^2\right)^{1/2}} \\
&= \frac{2L^2 - 4x^2}{\left(L^2 - x^2\right)^{1/2}}
\end{aligned}
$$

Since $x \in [0, L]$ we see that the denominator is always well defined (we don't take the square root of a negative numer). It is the case that the denominator evaluates to zero when $x = \pm L$ and so the derivative is undefined at these two points. The other critical points occur when the numerator is zero. This occurs exactly when $x = \pm L/\sqrt{2}$. So, there are a total of four critical points. Only two of them live in the interval $[0, L]$, specifically $x = L$ and $x = L/\sqrt{2}$. So, we need to check these critical points as well as the endpoints (note that one of the critical points *is* an endpoint, so we really only need to check a total of three values).

$$
\begin{aligned}
A(0) &= 2(0)\sqrt{L^2 - 0^2} = 0 \\
A(L/\sqrt{2}) &= 2(L/\sqrt{2})\sqrt{L^2 - L^2/2} = 2(L/\sqrt{2})\sqrt{L^2/2} = 2(L/\sqrt{2})(L/\sqrt{2}) = L^2 \\
A(L) &= 2(L)\sqrt{L^2 - L^2} = 0
\end{aligned}
$$

So, we see that the maximum area occurs when $x = L/\sqrt{2}$ and $y = 2L/\sqrt{2}$ giving us an area of L^2.

§ 3.6 # 45) A small island is 2km off shore in a large lake. A woman on the island can row her boat 10km/h and can run at a speed of 20km/h. If she rows to the closest point of the straight shore, she will land 6km from a village on the shore. Where should she land to reach the village most quickly by a combination of rowing and running?

Solution : First we need to sketch a figure to see what is going on.

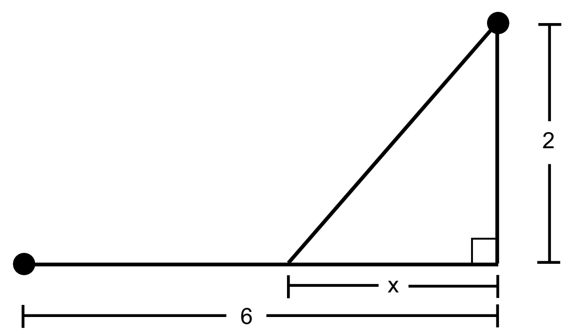

Figure 8: Here we see the possible path the woman may take on her way to the village. The point on the tip of the triangle represents her start point on the island. If she were to go straight to shore then she would travel $2km$ across the water as indicated by the vertical line in the illustration. Once she hits land I have decided to let x denote the length of land that she skipped. I did this because I know that at some point I will need to compute the length of the longest side of the triangle, and this choice of x will make that computation much easier.

We need to minimize the time it takes to get from the island to the village. Therefore, we first need to find an equation that tell us how much time it will take depending on her choice of path. So, first we need to figure out how much time it will take for each part of her path. As seen in the illustration the distance the woman will travel across water is the length of the hypotenuse of the right triangle. Thus this distance will be $\sqrt{4+x^2}$ by the Pythagorean theorem. Furthermore, the remaining distance she travels on land will be exactly $6-x$. Therefore, the total distance she travels is the sum of these two quantities. However, we are trying to minimize the time it takes. So, we know how fast she can travel across land and water, and we have an expression for the distance as well. So, since we have the equation $d = rt$ (distance equals the product of rate and time) we know that we can compute the time it takes her to cross each piece as well. Indeed, to find the time we need only solve the distance equation for t. This gives us $t = d/r$. Thus, to cross the hypotenuse it will take time $\frac{1}{10}\sqrt{4+x^2}$ and the time needed to cross the land will be $\frac{1}{20}(6-x)$. Then, if we add these two times together, we will obtain the function that we want to minimize.

$$T(x) = \frac{\sqrt{4+x^2}}{10} + \frac{(6-x)}{20}.$$

So, now we need to find a closed interval so restrict our attention to so that we may use the theory. This is alomost immediate from the picture. We see that the interval we want is exactly $[0, 6]$. Furthermore, we see that the function we have is continuous everywhere. Since the only term that looks a little dangerous is the square root term. However, this isn't a problem because $4 + x^2 > 0$ for all $x \in \mathbb{R}$. So, in fact, $T(x)$ is continuous everywhere, and so it is certainly continuous on the closed interval $[0, 6]$.

So, the theory tells us that we can find the minimum (and maximum) of $T(x)$ on the closed interval $[0.6]$ and that it is found by evaluating $T(x)$ at an endpoint of the closed interval, or at a critical point of $T(x)$. So, let's compute the derivative of $T(x)$.

$$\begin{aligned} T'(x) &= \frac{1}{10}\left(\frac{1}{2}(4+x^2)^{-\frac{1}{2}}(2x)\right) - \frac{1}{20} \\ &= \frac{x}{10\sqrt{4+x^2}} - \frac{1}{20} \\ &= \frac{2x - \sqrt{4+x^2}}{20\sqrt{4+x^2}} \end{aligned}$$

Notice that this will be defined everywhere because the denominator is strictly positive (i.e., never zero) and the numerator is defined everywhere for reasons similar to the above showing that $T(x)$ is continuous. So, the only critical points will occur when $T'(x) = 0$. This only happens when the numerator is zero. So, let's find when that is zero.

$$\begin{aligned} &2x - \sqrt{4+x^2} = 0 \\ \Rightarrow &2x = \sqrt{4+x^2} \\ \Rightarrow &4x^2 = 4 + x^2 \\ \Rightarrow &x^2 = \frac{4}{3} \\ \Rightarrow &x = \pm\frac{2}{\sqrt{3}} \end{aligned}$$

We know that we are only concerned with the positive solution since the other is outside of our interval. So, it remains to check the value of $T(x)$ and the endpoints and the critical point.

$$
\begin{aligned}
T(0) &= \frac{1}{2} \\
T\left(\frac{2}{\sqrt{3}}\right) &= \frac{\sqrt{3}+3}{10} \approx .4732050808 \\
T(6) &= \frac{\sqrt{10}}{5} \approx .6324555320
\end{aligned}
$$

So, we se that the route that requires the least time is the one corresponding to $x = 2/\sqrt{3}$, and the time this will take is approximately .4732050808 hours.

§ 3.6 #27) A printing company has eight presses, each of which can print 3600 copies per hour. It costs \$5.00 to set up each press for a run and $10 + 6n$ dollars to run n presses for 1 hour. How many presses should be used to print 50,000 copies of a poster most profitably?

Solution : We want to maximize the profit. To maximize the profit obtained by selling 50,000 posters will be achieved if we can *minimize* the cost to produce these 50,000 posters. So, we ned to construct a function to minimize. Let n be the number of presses that we will use. Then, if t is the time it takes to print 50,000 copies we see that to find the time in terms of the number of presses we use we need to solve the equation $3600nt = 50,000$ (we get this equation by noticing that if I have n presses then they can produce $3600n$ copies in an hour. So they can produce $3600nt$ copies in t hours. Finally, if I know that I need to make 50,000 copies, then I need to solve $3600nt = 50,000$ for t). Solving this tells us that,

$$t = \frac{50000}{3600n} = \frac{125}{9n}.$$

Then, now that we know the time we can compute the cost. We want the cost to run n presses for t hours. This is given by,

$$c(n) = 5n + (10 + 6n)t = 5n + (10 + 6n)\frac{125}{9n} = 5n + \frac{1250}{9n} + \frac{750}{9}.$$

We know that we can not use any less than 0 presses and no more than 8. So it seems that the interval we are interested in is $[0, 8]$. But wait, STOP THE PRESSES! ***The cost function is not continuous on this interval!*** It's not even defined for $n = 0$. So, let's ponder for a minute. Can we get rid of the zero somehow? That is the only place where $c(n)$ is discontinuous. Indeed we can! We aren't going to be able to get 50,000 copies of anything if we don't have at least one press. So, let's use the closed interval $[1, 8]$ instead. Then the function is continuous on this interval and we can apply the theory. We need to find the critical points.

$$c'(n) = 5 - \frac{1250}{9n^2} = \frac{45n^2 - 1250}{9n^2}$$

This is only undefined when $n = 0$ and this is not in our new and improved closed interval. So, the critical points can only be where $c'(n) = 0$. So, we need to solve for that.

$$
\begin{aligned}
&c'(n) = 0 \\
\Rightarrow &45n^2 - 1250 = 0 \\
\Rightarrow &n^2 = \frac{1250}{45} \\
\Rightarrow &n^2 = \frac{250}{9} \\
\Rightarrow &n = \frac{5\sqrt{10}}{3} \approx 5.270462768
\end{aligned}
$$

Now, before we get ahead of ourselves let's stop and think for a minute. What is this critical point we just found correspond to? It's suppose to be the number of presses we want to use right? So, does it make sense to use 5.270462768 presses? Not really. How can you use a fraction of a press? So, it seems to reason that we will want to use some whole number near 5.270462768. So, lets try 5 and 6. Then we have,

$$
\begin{aligned}
c(1) &= \frac{2045}{9} \approx 227.2222222 \\
c(5) &= \frac{1225}{9} \approx 136.1111111 \\
c\left(\frac{5\sqrt{10}}{3}\right) &= \frac{50\sqrt{10} + 250}{3} \approx 136.0379610 \text{ (just for kicks)} \\
c(6) &= \frac{3685}{27} \approx 136.4814815 \\
c(8) &= \frac{5065}{36} \approx 140.6944444
\end{aligned}
$$

From the above data we see that we minimize the cost to print the posters when we use 5 presses. Just to see that this looks right, look at the graph of the cost function below.

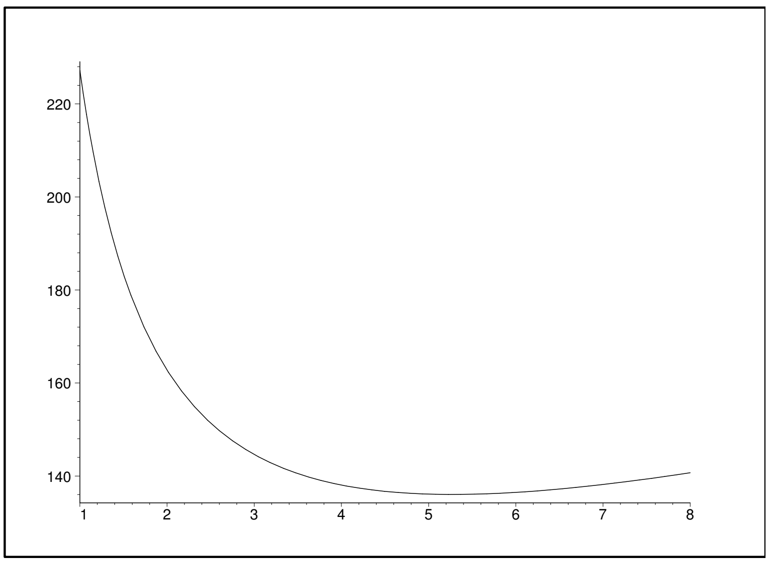

Figure 9: A plot of the cost function $c(n)$