Name:____________________

Test 1 Solutions
Fall 2002
Math 2200 MWF 9:05-9:55am
September 13, 2002

Directions : You have 50 minutes to complete all 5 problems on this exam. There are a possible 100 points to be earned on this exam. You may not use programmable calculators. However, you may use scientific calculators if you wish. Please be sure to show all pertinent work. *An answer with no work will receive very little credit!* If any portion of the exam is unclear please come to me and I will elaborate provided I can do so without giving away the problem.

1. (30 points)
   Compute the following limits.

   (a) $\lim_{x\to 0} \dfrac{x^4+12x+1}{x+1}$ (b) $\lim_{x\to 3} \dfrac{x^2+x-12}{x^2-2x-3}$ (c) $\lim_{\theta\to 0} \dfrac{\sin 6\theta}{3\theta}$

   Solution :

   a) Since the denominator does not approach zero as x appraoaches zero we can use the quotient law.

$$\lim_{x\to 0} \frac{x^4+12x+1}{x+1} = \frac{0^4+12(0)+1}{0+1} = 1$$

   b) We can not immediately use the quotient law since the denominator limits to zero, as does the numerator. We can, however, simplify the expression by factoring and then use the quotient law.

$$\lim_{x\to 3} \frac{x^2+x-12}{x^2-2x-3} = \lim_{x\to 3} \frac{(x+4)(x-3)}{(x+1)(x-3)} = \lim_{x\to 3} \frac{x+4}{x-3} = \frac{7}{4}.$$

   c) We will need to use the fact that $\lim_{\theta\to 0} \sin\theta/\theta = 1$ to solve this problem.

$$\lim_{\theta\to 0} \frac{\sin 6\theta}{3\theta} = \lim_{\theta\to 0} \frac{\sin 6\theta}{3\theta}\frac{2}{2} = \lim_{\theta\to 0} \frac{2\sin 6\theta}{6\theta} = 2\lim_{\theta\to 0} \frac{\sin 6\theta}{6\theta} = 2\cdot 1 = 1.$$

2. (10 points)
Consider the function

$$f(x) = \begin{cases} \sin(x) & x \leq 0 \\ x & x > 0 \end{cases}.$$

Is $f(x)$ continuous on $\mathbb{R}$? Explain your answer.

Solution : Since $\sin x$ and the function x are continuous on all of $\mathbb{R}$ we know that there will be no problem when $x \neq 0$. There is potential for disaster at $x = 0$ because we are attempting to glue two different functions together at this point. So, we need to make sure that the definition of continuity is satisfied at $x = 0$. Then we can assert that $f(x)$ is continuous on all of $\mathbb{R}$. So, to do this we need to show that the function is defined at $x = 0$, the left hand limit at zero exists, the right hand limit at zero exists, the left and right limits agree, and the value of the left and right limits is exactly $f(0)$. So, by definition we see that $f(x)$ is defined at $x = 0$ and that $f(0) = \sin 0 = 0$. Furthermore,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \sin x = 0, \text{ and } \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0.$$

So, not only do the left and right limits exist, but they are each equal to zero. Since $f(0) = 0$ we deduce that f is continuous at $x = 0$. Therefore, we know that $f(x)$ is continuous for all real numbers.

3. (15 points)

Is the following function continuous on $\mathbb{R}$? If not, where is it discontinuous, why is it discontinuous there, and can you find a function that is continuous on $\mathbb{R}$ and that agrees with f where f is continuous?

$$f(x) = \frac{x^4 - 6x^2}{|x|}$$

Solution : No, the function is not continuous at every real number. The function is not defined at $x = 0$ so it can not be continuous there. It is, however, continuous everywhere else. To see this we need to rewrite $f(x)$ using the definition of absolute value.

$$f(x) = \frac{x^4 - 6x^2}{|x|}$$

$$= \begin{cases} \frac{x^4-6x^2}{x} & x > 0 \\ \frac{x^4-6x^2}{-x} & x < 0 \end{cases}$$

$$= \begin{cases} x^3 - 6x & x > 0 \\ -x^3 + 6x & x < 0 \end{cases} .$$

So, we can see that when $x < 0$ $f(x)$ is just a polynomial and polynomials are continuous everywhere, so $f(x)$ is continuous for all $x < 0$. Likewise, if $x > 0$ then $f(x)$ is a polynomial (only different by a sign) and so $f(x)$ is continuous for all $x > 0$ as well.

In order to find a continuous function that agrees with f everywhere f is continuous it must look like,

$$g(x) = \begin{cases} f(x) & x \neq 0 \\ ? & x = 0 \end{cases} .$$

So, we only need to find one value so that we can fill the hole. To do this we will need to use the definition of continuity. We want $g(x)$ to be continuous. We know that it is continuous whenever $x \neq 0$ because $f(x)$ is continuous whenever $x \neq 0$. So, if $g(x)$ is to be continuous at $x = 0$ we know that $g(0) = \lim_{x\to 0} g(x)$. So, to fill in the hole we have,

$$g(x) = \begin{cases} f(x) & x \neq 0 \\ \lim_{x\to 0} g(x) & x = 0 \end{cases} .$$

Where,

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} -x^3 + 6x = 0, \text{ and } \lim x \to 0^+ g(x) = \lim_{x\to 0^+} x^3 - 6x = 0.$$

Since the left limit and the right limit exist and are equal we know that we can define $g(x)$ so that it is continuous by

$$g(x) = \begin{cases} f(x) & x \neq 0 \\ 0 & x = 0 \end{cases} .$$

4. (20 points)
Show that there is a point $x \in [0, 1]$ so that $\cos \pi x = x^{562}$.

Solution : Hellooo intermediate value theorem. Let $f(x) = \cos \pi x - x^{562}$. $f(x)$ is continuous since $\cos \pi x$ is a composition of continuous functions, x^{562} is a continuous polynomial, and $f(x)$ is their difference. Moreover, $f(0) = 1 - 0 = 1$ and $f(1) = -1 - 1 = -2$. Since f is continuous on $[0, 1]$, $f(0) > 0$, and $f(1) < 0$ we know, by the intermediate value theorem, that there is a point c in between 0 and 1 so that $f(c) = 0$. Thus, $\cos \pi c - c^{562} = 0$ which implies $\cos \pi c = c^{562}$ as desired.

5. (25 points)

(a) State the definition of the derivative. (5/25 points)

(b) Using the definition, compute the derivative of the function $f(x) = \sqrt{x}$ for all $x > 0$. (15/25 points)

(c) Write down the equation of the tangent line to the graph $y = f(x)$ at the point $x = 4$. (5/25 points)

Solution :

a) The derviative of $f(x)$ at the point a is the slope of the tangent line to the graph of $f(x)$ at the point $(a, f(a))$ and is defined to be

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided that this limit exists. Moreover, if $f'(a)$ exists for each a in the domain of $f(x)$, then we say that $f(x)$ is differentiable on its domain and the derivative of $f(x)$ is the function $f'(x)$.

b) Here we go,

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \right) \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\
&= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}
\end{aligned}$$

c) To find the equation of the line tangent to the graph $y = f(x)$ at $x = 4$ we will need to pieces of information. It suffices to find any point on the line as well as the slope of the line. Since we know that ther line is to be tangent to the graph $y = f(x)$ at $x = 4$, we know that the slope must be equal to $f'(4) = 1/(2\sqrt{4}) = 1/4$. Then, we also know that the point $(4, f(4))$ must be on the line and so we have enough information to use the point slope formula. Indeed, the equation of the line will be,

$$y - f(4) = f'(4)(x - 4)$$

which reduces to

$$y = \frac{1}{4}x + 1.$$

Bonus :

(10 points)

Show (by sketching a graph) that there exists a discontinuous function f so that for each $a \in \mathbb{R}$

$$f(a) \text{ exists, } \lim_{t \to a^-} f(x) \text{ exists, and } \lim_{t \to a^+} f(x) = f(a).$$

Solution : Such a function is the following,

$$f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

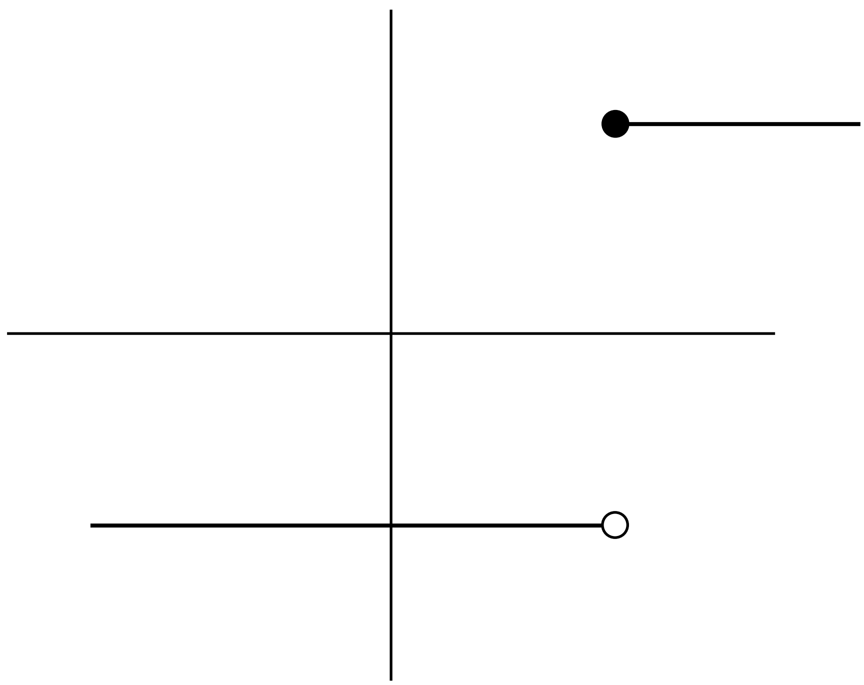

Figure 1: Sketch of $f(x)$