Name:____________________

Test 4
Fall 2002
Math 2200 MWF 9:05-9:55am
November 15, 2002

Directions : You have 50 minutes to complete all 5 problems on this exam. There are a possible 100 points to be earned on this exam. You may not use programmable calculators; however you may use scientific calculators if you wish. Please be sure to show all pertinent work. *An answer with no work will receive very little credit!* If any portion of the exam is unclear please come to me and I will elaborate provided I can do so without giving away the problem.

1. (20 points)
Use implicit differentiation to compute equation of the line tangent to the curve described by

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 5$$

at the point (8,1).

Solution : We only need to find the slope at the point $(8, 1)$.

$$\frac{d}{dx}\left(x^{\frac{2}{3}} + y^{\frac{2}{3}}\right) = \frac{d}{dx}5$$

$$\Rightarrow \frac{2}{3}x^{-\frac{1}{3}}\frac{dx}{dx} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\frac{2}{3}x^{-\frac{1}{3}}}{\frac{2}{3}y^{-\frac{1}{3}}}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$

So, at the point $(8, 1)$ the slope will be $-1^{1/3}/8^{1/3} = -1/2$. Thus the tangent line is,

$$y = -\frac{1}{2}(x - 8) + 1 = -\frac{1}{2}x + 5.$$

2. (20 points)
Luke and his sister Leia each begin piloting a ship away from the same port at noon. Luke is traveling due east at a constant rate of 5 mi/h while his sister is traveling due north at a constant rate 7 mi/h. What is the rate of change of the distance between Luke and Leia exactly two hours after they left port?

Solution : First we draw a picture.

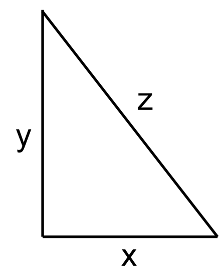


Figure 1: Leia is heading north along the edge labeled y and luke is traveling east on the edge labeled x.

We know that at $2:00$ $dx/dt = 5$ and $dy/dt = 7$ and we would like to find dz/dt. The equation $x^2 + y^2 = z^2$ relates all of these values, so we will take an implicit derivative.

$$\frac{d}{dt}\left(x^2 + y^2\right) = \frac{d}{dt}z^2$$

$$\Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

It looks like we will need some more information to solve this problem. We will need to know x, y, and z. But we know that each ship has been traveling for two hours and that they are traveling at a constant rate. Therefore we see that $x = 10$, $y = 14$, and we can solve for z since $10^2 + 14^2 = z^2$ which tells us that $z = \sqrt{100 + 196} = \sqrt{296}$. We have all of the information we need so we can insert it in the previous equation.

$$\Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$\Rightarrow 2(10)(5) + 2(14)(7) = 2(\sqrt{296})\frac{dz}{dt}$$

$$\Rightarrow 100 + 196 = 2\sqrt{296}\frac{dz}{dt}$$

$$\Rightarrow \frac{148}{\sqrt{296}} = \frac{dz}{dt}.$$

3. (20 points)
   Use linear approximation to approximate $\sqrt{(37)}$.

   Solution : We want to find the equation of a tangent line to the function $f(x) = \sqrt{(x)}$ at some point $x = a$ near the point $x = 37$. We know that the equation of the tangent line will be

   $$L(x) = f'(a)(x - a) + f(a)$$

   and so we had better choose a so that we can evaluate $f'(a)$ and $f(a)$. Note that 37 is pretty close to 36 and we know how to compute $\sqrt{36}$. So lets use $a = 36$. We then know that the tangent line will be

   $$L(x) = -\frac{1}{2\sqrt{36}}(x - 36) + \sqrt{36} = \frac{1}{12}x + 3.$$

   So, $f(37) \approx L(37) = \frac{37}{12} + 3 = \frac{73}{12} \approx 6.083333333$.

4. (20 points)
Let
$$f(x) = \frac{1}{5}x^5 + \frac{3}{4}x^4 + \frac{2}{3}x^3.$$

(a) Find the intervals on which $f(x)$ is increasing and the intervals on which $f(x)$ is decreasing.

(b) Classify all critical points of $f(x)$ as local minima, local maxima, global minima, global maxima, or none of these.

(c) Sketch the graph of the function.

Solution :

(a) To find the intervals we need to find where the derivative is positive and where it is negative.

$$f'(x) = x^4 + 3x^3 + 2x^2 = x^2(x^2 + 3x + 2) = x^2(x + 1)(x + 2)$$

So, there are critical points at $x = 0$, $x = -1$, and $x = -2$. Since we know that the derivative can only possibly change sign at these points it suffices to determine the sign of $f'(x)$ when $x = -3$, $x = -3/2$, $x = -1/2$, and $x = 1$.

$$f'(-3) = (-3)^2(-3 + 1)(-3 + 2) = 9(-2)(-1) = 18 > 0$$
$$f'(-\frac{3}{2}) = \left(-\frac{3}{2}\right)^2 \left(-\frac{3}{2}) + 1\right) \left(-\frac{3}{2} + 2\right) = \left(\frac{9}{4}\right) \left(-\frac{1}{2}\right) \left(\frac{1}{2}\right) = -\frac{9}{16} < 0$$
$$f'(-\frac{1}{2}) = \left(-\frac{1}{2}\right)^2 \left(-\frac{1}{2} + 1\right) \left(-\frac{1}{2} + 2\right) = \left(\frac{1}{4}\right) \left(\frac{1}{2}\right) \left(\frac{3}{2}\right) = \frac{3}{16} > 0$$
$$f'(1) = (1)^2 (1 + 1) (1 + 2) = (1) (2) (3) = 6 > 0$$

We deduce that $f(x)$ is increasing on the intervals $(-\infty, -2)$, $(-1, 0)$, and $(0, \infty)$ and $f(x)$ is decreasing on the interval $(-2, -1)$.

(b) $f(x)$ has a local maximum at $x = -2$, a local minimum at $x = -1$, and neither at $x = 0$.

(c)

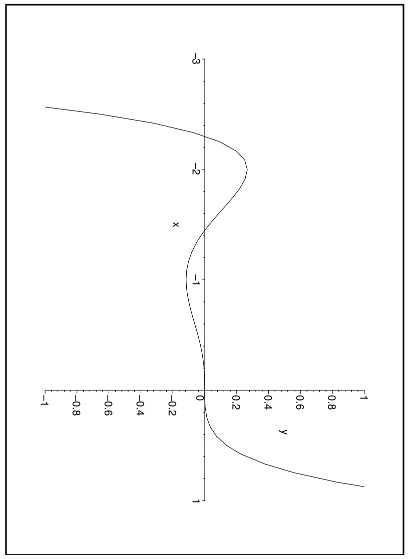
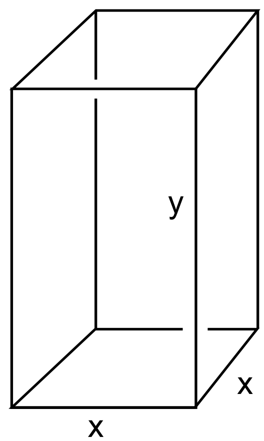


Figure 2: Here is Maple's plot.

5. (20 points)
A closed rectangular container with a square base is to have a volume of 2000 cm^3. It costs twice as much per square centimeter for the top and bottom as it does for the sides. Find the dimensions of the container of least cost.

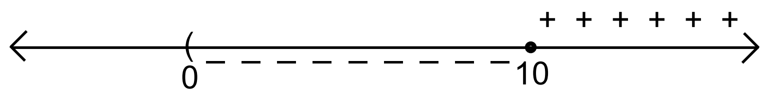
Solution : Let's start with a picture.



We know that the volume of this guy is $V = 2000 = x^2y$. Now we need to come up with a cost function to minimize. It should have something to do with the surface area which is $S = 2x^2 + 4xy$. Indeed, we know that the material for the top and bottom is to cost twice that of the material for the sides. So, we have the cost equation as $C = 2(2x^2) + 4xy = 4x^2 + 4xy$. Using the volume information we see that we can write $y = 2000/x^2$ and so our cost function can be written as a function of one variable, $C(x) = 4x^2 + 4x(2000/x^2) = 4x^2 + 8000/x$. Notice that x is a physical dimesnsion, so we know that it cannot be negative. Furthermore, as x approaches zero notice that y approaches infinity. That is y is undefined if $x = 0$ since $y = 2000/x^2$. Also, if y approaches zero we see that x must approach infinity to compensate. Thus, we see that the domain of interest for this problem is the open interval $(0, \infty)$. The next step is to compute a derivative and find the critical points.

$$C'(x) = 8x - \frac{8000}{x^2} = \frac{8x^3 - 8000}{x^2}$$

The critical points occur when $C'(x) = 0$ or when $C'(x)$ is undefined. $C'(x)$ is undefined only when $x = 0$ and this is not in our domain of interest so we can ignore it. $C'(x) = 0$ exactly when the numerator is equal to zero. That is $C'(x) = 0 \iff 8x^3 - 8000 = 0 \iff x^3 = 1000 \iff x = 10$. So, we only have one critical point. To see that it is going to give us the desired global minimum we can use the first derivative test. Observe that we only need to evaluate $C'(x)$ at some point in between 0 and 10 and in at some point larger than 10 since the only place it can possibly change sign in our domain of interest $(0, \infty)$ is when $x = 10$. Notice that $C'(1) = (8 - 8000)/1 = -7992 < 0$ and $C'(11) = (8(11^3) - 8000)/11^2 = (10648 - 8000)/121 > 0$.



So, looking at the signs we can see that this value does indeed give us minimum. The dimensions that minimize cost are $x = 10$ and $y = 20$ giving us a cost of $C(10) = 4(10^2) + 8000/10 = 1200$.