Here are a couple of proofs to the following

Theorem 1. *Given a set of $n+1$ positive integers, none exceeding 2n, there is at least one integer in this set that divides another integer in the set.*

Proof. (Induction) Let $P(n)$ be the proposition "if A is a set of $n+1$ positive integers, none exceeding $2n$, then there is at least one integer in A that divides another integer in A," and let $S = \{n \in \mathbb{N} | P(n) \equiv T\}$. We will use PMI (the Principle of Mathematical Induction) to show that $S = \mathbb{N}$. To do this, we need to establish two cases.

(1) $1 \in S$. (The Base Case)

(2) If $n \in S$, then $n+1 \in S$. (The Induction Step)

To see the first condition, notice that when $n = 1$ the set A can only consist of 2 positive integers no larger than 2. Since there are exactly two such integers it follows that $A = \{1, 2\}$. Notice that $1 \mid 2$ and so the base case has been established.

Now we define the induction hypothesis. Assume that $P(n)$ is true. That is, assume that if A is any set containing $n+1$ positive integers, none of which exceed $2n$, then there exist elements $a, b \in A$ so that $a \mid b$. We need to use this to prove that $P(n+1)$ is true.

$P(n+1)$ is the statement "if B is a set of $(n+1)+1$ positive integers, none exceeding $2(n+1)$, then there is at least one integer in B that divides another integer in B." We will attempt to prove this directly using the induction hypothesis. So, assume that B is a set of $n+2$ positive integers none of which exceed $2n+2$. We need to break this up into a few cases.

Case 1: If $2n+1 \notin B$ and $2n+2 \notin B$, then every element of B is less than or equal to $2n$. So, take out any one element x from B. What we have left is the set $B \setminus \{x\}$ which contains $n+1$ positive integers none of which exceed $2n$. So, by the induction hypothesis, there exist elements $a, b \in B \setminus \{x\}$ so that $a \mid b$. Note that a and b are also in B. So we have found two elements of B so that one divides the other and we are done.

Case 2: If $2n+1 \in B$ or $2n+2 \in B$ but not both. This case is really the same as the last case with one little change. We know that either $2n+1$ or $2n+2$ are in B but not both. So, it follows that every element of B is less than or equal to $2n$ with exactly one exception ($2n+1$ or $2n+2$, whichever is in B). So, take out the exception. What remains is a set of $n+1$ elements none of which exceed $2n$. So, as in the last case, the induction hypothesis guarantees us two elements a and b so that $a \mid b$. Since these elements must be in B we are done.

Case 3: If $2n+1 \in B$ and $2n+2 \in B$, then we have a little problem. We are only allowed one element larger than $2n$ in B if we want to use the same argument as above. If I throw out $2n+1$, then I have a set $B \setminus \{2n+1\}$ consisting of $n+1$ positive integers but I don't know that none of them exceed $2n$. In fact there *is* an element that exceeds $2n$, specifically $2n+2 > 2n$. So, we cannot immediately impose the induction hypothesis. So, we need to use a nontrivial trick. Consider the set $B \setminus \{2n+2\}$ and add the element

$n+1$ provided $n+1 \notin B$ (if $n+1 \in B$ then we are done since $n+1 \mid 2n+2$). This gives us some new set $C = (B \setminus \{2n+2\}) \cup \{n+1\}$. That is, we have thrown out the element $2n+2$ and replaced it with the element $n+1$. We need to check that this does not invalidate the problem. Notice that this trick will be dangerous if $n+1$ is divided by some element of B that $2n+2$ is not divided by *or* if $n+1$ divides some element of B. If either of these things happen then I have altered the set in such a way that the element I have added would make the statement true, but without it the statement may be false. Not to worry. If there is an element $a \in B$ that divides $n+1$, then it must also divide $2n+2 = 2(n+1)$. Furthermore, since $2n+2 = 2(n+1)$ we see that $2n+2$ is the *smallest* positive integer multiple of $n+1$ (other than $(n+1)(1)$ which we don't consider because a set cannot have repeated elements). But, we threw out $2n+2$, so there is no element of C that can possibly be divided by $n+1$. It follows that we have not changed the validity of our proof by removing $2n+2$ and adding $n+1$ in its place. Now, we have a set C that contains $n+2$ positive integers and exactly one of which exceeds $2n$. We have reduced this case to the previous case and so we are done. $\square$

Proof. (Pigeon Hole Principle) Copied verbatim from example 11 in § 4.2 of Kenneth H. Rosen's Discrete Mathematics and Its Applications, Fifth Edition, pg 317.

Write each of the $n+1$ integers $a_1, a_2, \ldots, a_{n+1}$ as a power of 2 times an odd integer. In other words, let $a_j = 2^{k_j} q_j$ for $j = 1, 2, \ldots, n+1$, where k_j is a nonnegative integer and q_j is odd. The integers $q_1, q_2, \ldots, q_{n+1}$ are all odd positive integers less than $2n$. Since there are only n odd positive integers less than $2n$, it follows from the pigeonhole principle that two of the integers $q_1, q_2, \ldots, q_{n+1}$ must be equal. Therefore, there are integers i and j such that $q_i = q_j$. Let q be the common value of q_i and q_j. Then, $a_i = 2^{k_i} q$ and $a_j = 2^{k_j} q$. It follows that if $k_i < k_j$, then a_i divides a_j; while if $k_i > k_j$, then a_j divides a_i. $\square$