MATH 2610 Spring 2003
Quiz 2

1. Show that if $A = \{a_1, a_2, a_3, \ldots\}$ is a countably infinite set and if x is any element not in A, then the set $B = \{x, a_1, a_2, a_3, \ldots\}$ is also countably infinite.

   Solution : Since A is countably infinite we know that there exists a bijection $f : \{a_1, a_2, a_3, \ldots\} \longrightarrow \mathbb{N}$. Without loss of generality lets suppose that this function is defined by $f(a_i) = i$. Then, define a new function $g : \{x, a_1, a_2, \ldots\} \longrightarrow \mathbb{N}$ by the following:

$$g(y) = \begin{cases} f(y) + 1 & \text{if } y \neq x \\ 1 & \text{if } y = x. \end{cases}$$

   Then g is injective since it is merely a shift of an inective function and it is onto since f was onto and we "filled in the hole" obtained by shifting f to the right one unit by defining $g(x) = 1$. It follows that g is a bijection and so the set $\{x, a_1, a_2, \ldots\}$ is countably infinite.

2. Show that if $a, b, c, d \in \mathbb{Z}$ such that $a \mid c$ and $b \mid d$, then $ab \mid cd$.

   Solution : We will prove this directly. Assume that $a, b, c, d \in \mathbb{Z}$ so that $a \mid c$ and $b \mid d$. Then we know that there exist integers s and t so that $c = as$ and $d = bt$ by definition of $a \mid c$ and $b \mid d$. Then,

$$\begin{aligned} cd &= (as)(bt) \\ &= (ab)(st). \end{aligned}$$

   So, we have found an integer st so that $cd = (ab)(st)$. That is $ab \mid cd$ as desired.

3. List five integers that are congruent to 4 modulo 12.

   Solution : All integers congruent to 4 modulo 12 look like $4 + k \cdot 12$, where $k \in \mathbb{Z}$. Pick any 5 values for k say, $k = 1, 2, 3, 4, 5$. This gives us the numbers 16, 28, 40, 52, and 64, each of which is congruent to 4 modulo 12.