MATH 2610 Spring 2003
Quiz 3

1. Use mathematical induction to show that $2 + 4 + 6 + \cdots + 2n = n(n + 1)$.

   Solution : Let $S = \{n \in \mathbb{N} | 2 + 4 + 6 + \cdots + 2n = n(n + 1)\}$. We will use the principle of mathematical induction to show that $S = \mathbb{N}$, hence the statement is true for all natural numbers. PMI (the Principle of Mathematical Induction) says that we only need to show two things. Specifically,

   1) $1 \in S$ (Base Case)

   2) If $n \in S$, then $n + 1 \in S$. (Induction Step)

   So, lets establish the base case. If $n = 1$, then our sum (the left side of the equation) just looks like 2. The right side of the equation is $1(1 + 1) = 2$. So, the left side is equal to the right side and we have established the base case. It remains to show the induction step. We begin by making the induction hypothesis (IHOP). Assume $2 + 4 + 6 + \cdots + 2n = n(n + 1)$. We will try to use this assumption to show $LS = 2+4+6+\cdots+2n+2(n+1) = (n + 1)((n + 1) + 1) = RS$. We have,

$$\begin{aligned} LS &= 2 + 4 + 6 + \cdots + 2n + 2(n + 1) \\ &= (2 + 4 + 6 + \cdots + 2n) + 2(n + 1) \\ &= (n(n + 1)) + 2(n + 1) \text{ (by IHOP)} \\ &= n^2 + n + 2n + 2 \\ &= n^2 + 3n + 2. \end{aligned}$$

   Notice, finally, that $RS = (n + 1)((n + 1) + 1) = (n + 1)(n + 2) = n^2 + 3n + 2$, so we have shown that $LS = RS$. So, by PMI, $S = \mathbb{N}$. I.e., $2 + 4 + 6 + \cdots 2n = n(n + 1)$ for all $n \in S = \mathbb{N}$.

2. How many bit strings of length n, where $n \in \mathbb{N}$, start and end with 1's?

   Solution : Since we are fixing the first and last position to be 1's, this restricts us to only being able to make $n-2$ choices. This is exactly equivalent to choosing a bit string of length $n - 2$, which can be done in 2^{n-2} ways.

3. How many license plates can be made using either 3 digits followed by 3 capital letters, or 3 capital letters followed by 3 digits.

   Solution : Suppose that we have the letters first. Then we have 26 possibilities for the first choice. For each of those choices we have 26 possible choices for the second character. For each of these two choices we have 26 choices for the third character. So far we seen a possible 26^3 choices. For each of those choices of three letters we need to choose 3 numbers (0-9). This accounts for 10^3 choices bring the grand total to $26^3 \cdot 10^3$ possible choices for a license plate starting with three letters followed by three numbers. If we swap this we see that there are $10^3 \cdot 26^3$ different ways to select a license plate with three numbers followed by three letters. The sum rule says that the total number of license plates that can be made using either 3 digits followed by 3 capital letters, or 3 capital letters followed by 3 digits is $26^3 \cdot 10^3 + 10^3 \cdot 26^3 = 2(26^3 \cdot 10^3)$.