Example 1: Find a solution to $a_n = 5a_{n-2} - 4a_{n-4}$ with $a_0 = 3$, $a_1 = 2$, $a_2 = 6$, and $a_3 = 8$.

Solution : Recall in class that we showed the characteristic polynomial factors as,

$$r^4 - 5r^2 + 4 = (r^2 - 4)(r^2 - 1) = (r + 2)(r - 2)(r + 1)(r - 2).$$

So, since there are four distinct roots of the characteristic of our degree four linear homogeneous constant coefficient recurrence relation, we know that the general solution will be,

$$a_n = \alpha_1(1^n) + \alpha_2(-1)^n + \alpha_3(2^n) + \alpha_4(-2)^n.$$

To find the *particular* solution, we will need to use all four of our initial conditions. These conditions give us the following four equations,

$$\begin{aligned}
a_0 &= 3 = \alpha_1(1^0) + \alpha_2(-1)^0 + \alpha_3(2^0) + \alpha_4(-2)^0 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \\
a_1 &= 2 = \alpha_1(1^1) + \alpha_2(-1)^1 + \alpha_3(2^1) + \alpha_4(-2)^1 = \alpha_1 - \alpha_2 + 2\alpha_3 - 2\alpha_4 \\
a_2 &= 6 = \alpha_1(1^2) + \alpha_2(-1)^2 + \alpha_3(2^2) + \alpha_4(-2)^2 = \alpha_1 + \alpha_2 + 4\alpha_3 + 4\alpha_4 \\
a_3 &= 8 = \alpha_1(1^3) + \alpha_2(-1)^3 + \alpha_3(2^3) + \alpha_4(-2)^3 = \alpha_1 - \alpha_2 + 8\alpha_3 - 8\alpha_4.
\end{aligned}$$

Let's start solving. From the first equation we can solve for α_1 to obtain

$$\alpha_1 = 3 - \alpha_2 - \alpha_3 - \alpha_4. \tag{1}$$

Then, using this value for α_1 in the second equation and solving for α_2 we get,

$$\begin{aligned}
2 &= \alpha_1 - \alpha_2 + 2\alpha_3 - 2\alpha_4 \\
&= (3 - \alpha_2 - \alpha_3 - \alpha_4) - \alpha_2 + 2\alpha_3 - 2\alpha_4 \\
&= 3 - 2\alpha_2 + \alpha_3 - 3\alpha_4. \text{ So,} \\
\\
\alpha_2 &= \frac{1 + \alpha_3 - 3\alpha_4}{2}.
\end{aligned} \tag{2}$$

Continuing, we will insert the values we have solved for into the third equation and solve for α_3.

$$\begin{aligned}
6 &= \alpha_1 + \alpha_2 + 4\alpha_3 + 4\alpha_4 \\
&= (3 - \alpha_2 - \alpha_3 - \alpha_4) + \alpha_2 + 4\alpha_3 + 4\alpha_4 \\
&= 3 + 3\alpha_3 + 3\alpha_4. \text{ So,} \\
\\
\alpha_3 &= 1 - \alpha_4.
\end{aligned} \tag{3}$$

Finally, combining all of our information into the last equation, we can solve for α_4. Indeed,

$$\begin{aligned}
8 &= \alpha_1 - \alpha_2 + 8\alpha_3 - 8\alpha_4 \\
&= (3 - \alpha_2 - \alpha_3 - \alpha_4) - \alpha_2 + 8\alpha_3 - 8\alpha_4 \\
&= 3 - 2\alpha_2 + 7\alpha_3 - 9\alpha_4 \\
&= 3 - 2\left(\frac{1 + \alpha_3 - 3\alpha_4}{2}\right) + 7\alpha_3 - 9\alpha_4 \\
&= 3 - 1 - \alpha_3 + 3\alpha_4 + 7\alpha_3 - 9\alpha_4 \\
&= 2 + 6\alpha_3 - 6\alpha_4 \\
&= 2 + 6(1 - \alpha_4) - 6\alpha_4 \\
&= 8 - 12\alpha_4. \text{ So,} \\
\alpha_4 &= 0.
\end{aligned}$$

Substituting $\alpha_4 = 0$ back into equation (3) we see that $\alpha_3 = 1 - 0 = 1$. Continuing, using $\alpha_4 = 0$ and $\alpha_3 = 1$ in equation (2) gives us $\alpha_2 = (1+1-0)/2 = 1$. Finally, using $\alpha_2 = 1$, $\alpha_3 = 1$, and $\alpha_4 = 0$ in equation (1) we see that $\alpha_1 = 3 - \alpha_2 - \alpha_3 - \alpha_4 = 3 - 1 - 1 - 0 = 1$. Now we can write down the particular solution.

$$\begin{aligned}
a_n &= \alpha_1(1^n) + \alpha_2(-1)^n + \alpha_3(2^n) + \alpha_4(-2)^n \\
&= 1 + (-1)^n + 2^n.
\end{aligned}$$

Example 2: (#20 in section 6.2) Find the general form of the solutions of the recurrence relation $a_n = 8a_{n-2} - 16a_{n-4}$.

Solution : Mercifully, the problem is only asking for a general solution. So we will not need to do as much algebra as in the previous problem. The recurrence relation in question is certainly linear, with constant coefficients, and homogeneous. So, we need to fact the characteristic polynomial to see which of our theorems we can use. To do this, we need to know what the characteristic polynomial is. Recall, that when given the recurrence relation of the form,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

the characteristic will be,

$$p(r) = r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k.$$

So, with our problem we have a fourth degree recurrence relation of the form

$$a_n = 0a_{n-1} + 8a_{n-2} - 0a_{n-3} - 16a_{n-4} \text{ (Don't forget the 0 coefficients!)}.$$

The characteristic polynomial is then,

$$p(r) = r^4 - 0r^3 - 8r^2 - 0r - (-16) = r^4 - 8r^2 + 16.$$

We know that if there are any rational roots then they must be the quotient of some factor of 16 with some factor of 1. So, the possibilities are ± 1, ± 2, ± 4, ± 8, and ± 16. We can then just check each of these eight possibilities to find the roots using synthetic division (like I've been doing in class) or we can notice

$$\begin{aligned} p(r) &= r^4 - 8r^2 + 16 \\ &= (r^2)^2 - 8r^2 + 16 \\ &= (r^2 - 4)(r^2 - 4) \\ &= (r-2)(r+2)(r-2)(r+2) \\ &= (r-2)^2(r+2)^2. \end{aligned}$$

We see that there are two roots, 2 and -2, each with multiplicity 2. Therefore, the general solution will be,

$$a_n = (\alpha_{0,1} + \alpha_{0,2} n)2^n + (\alpha_{1,1} + \alpha_{1,2} n)(-2)^n$$

where $\alpha_{0,1}$, $\alpha_{0,2}$, $\alpha_{1,1}$, and $\alpha_{1,2}$ are constants dependent on the initial conditions.