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Test 1

Spring 2003

CS/MATH 2610

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Directions : You have 75 minutes to complete all 6 problems on this exam. There are a possible 100 points to be earned. You may not use your book or any notes. Please be sure to show all pertinent work. *An answer with no work will receive very little credit!* If any portion of the exam is unclear please come to me and I will elaborate provided I can do so without giving away the problem.

(1) (20 points)

Answer each of the following questions.

(a) Define a proposition.

A proposition is a declarative statement that is either true or false, but not both.

(b) What does it mean for two propositions P and Q to be logically equivalent?

Two propositions P and Q are equivalent provided they are either both true or both false. I.e., their truth tables agree.

(c) What is the negation of the statement $\forall x \exists y (P(x, y) \to Q(x, y))$?

$\forall x \exists y (P(x, y) \to Q(x, y)) \equiv \exists x \forall y (P(x, y) \wedge \sim Q(x, y))$

(d) Let A and B be sets. Define $A \times B$.

$A \times B = \{(a, b) \mid (a \in A) \wedge (b \in B)\}$, that is the set of all ordered pairs (a, b) where a is an element of A and b is an element of B.

(e) Let $A = \{1, 2, 3\}$ what is $\mathcal{P}(A)$? List the elements. ($\mathcal{P}(A)$ denotes the power set of A.)

$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

(2) (14 points)

Let P, Q, and R be propositions. Prove or disprove that $(P \to Q) \to R$ and $P \to (Q \to R)$ are logically equivalent.

Solution : This is false. We can show that it is false by either coming up with a counter example, or showing that the truth tables don't agree. Since I suspect that most folks went ahead and tried to use a truth table, that's what I will do.

P	Q	R	$P \to Q$	$Q \to R$	$(P \to Q) \to R$	$P \to (Q \to R)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	T	T	T	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T
F	F	F	T	T	F	T

Their truth tables don't agree, so they are not equivalent. Looking at the truth table also gives us the desired contradiction. Notice that,

$$(F \to T) \to F \equiv F \text{ but } F \to (T \to F) \equiv T.$$

For what it's worth, they only differ at one other place,

$$(F \to F) \to F \equiv F \text{ but } F \to (F \to F) \equiv T.$$

(3) (16 points)

Use Venn diagrams to sketch the following sets.

(a) $A \cap \left(\overline{B \cap C}\right)$

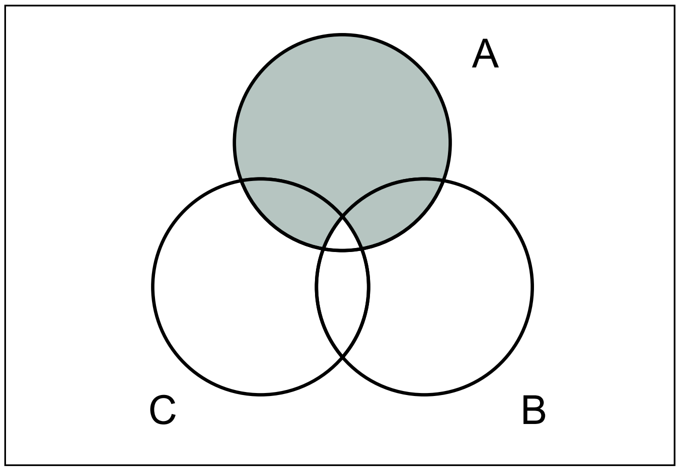

FIGURE 1. $A \cap (\overline{B \cap C})$

(b) $[(A \cap B) \cup (B \cap C) \cup (C \cap A)] - (A \cap B \cap C)$

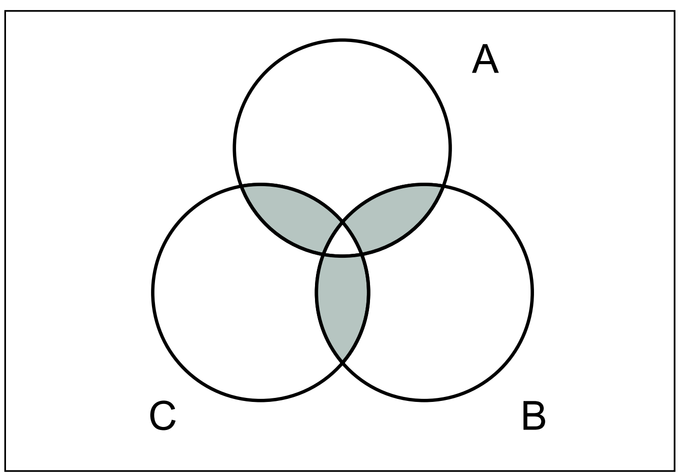

FIGURE 2. $[(A \cap B) \cup (B \cap C) \cup (C \cap A)] - (A \cap B \cap C)$

(4) (15 points)
Let n be an integer. Prove that if $7n+2$ is even, then n is even.

You will probably observe that a direct approach is not too fruitful. Let's see what happens if we try it. Assume that $7n+2$ is even and we want to show then that n is even. Since $7n+2$ is even we know there exists some integer k so that $7n+2=2k$. So, we try to solve for n to show that it is also some multiple of 2. We obtain $n=2[(k-1)/7]$. We have no idea that this is an integer! What if $k-1$ isn't divisible by 7? So, this approach will not work. Let's try proof by contradiction.

Proof. Suppose that $7n+2$ is even and that n is odd (here we negated the $p \to q$ statement to obtain $p \wedge \sim q$ which we will show is always false). Since n is odd we know that there is an integer k so that $n=2k+1$. Then computation shows us,

$$7n+2=7(2k+1)+2=14k+9=2(7k+4)+1.$$

Since $7k+4$ is an integer we deduce that $7n+2$ is odd, a contradiction. So our negated statement is always false. Hence, its negation is always true. So, If $7n+2$ is even, then n is even. (We showed that $\sim(p \to q) \equiv p \wedge \sim q \equiv F$, therefore $(p \to q) \equiv \sim(\sim(p \to q)) \equiv \sim F \equiv T$.) This completes the proof.

□

Proof by method of contrapositve works just as well. Remember that the statement $p \to q$ is logically equivalent to $\sim q \to \sim p$. So, lets try to prove directly the contrapositive statement "if n is odd, then $7n+2$ is odd."

Proof. Assume that n is odd. Then there exists a $k \in \mathbb{Z}$ so that $n=2k+1$. Then, direct computation shows that

$$7n+2=7(2k+1)+2=14k+9=2(7k+4)+1.$$

Since $l=7k+4$ is an integer and $7n+2=2l+1$ we have shown that $7n+2$ is odd. So, the contrapositive statement must be true. This completes the proof. □

(5) (20 points)

Let A, B, and C be sets. Prove that $A\cap\left(\overline{B\cap C}\right) = \left(A\cap\overline{B}\right)\cup\left(A\cap\overline{C}\right)$ by showing each side is a subset of the other side.(Just drawing a Venn diagram is not a proof.)

Proof. We already have a picture of $A\cap\left(\overline{B\cap C}\right)$ and so drawing the Venn diagram for the other side should make you believe that the statement is true. To prove it we need to show

(1) $A\cap\left(\overline{B\cap C}\right)\subseteq\left(A\cap\overline{B}\right)\cup\left(A\cap\overline{C}\right)$ and

(2) $\left(A\cap\overline{B}\right)\cup\left(A\cap\overline{C}\right)\subseteq A\cap\left(\overline{B\cap C}\right)$.

So, lets start with proving (1). Let $x\in A\cap\overline{(B\cap C)}$. Then we know that $x\in A$ and $x\in\overline{(B\cap C)}$. Since $x\in\overline{(B\cap C)}$ we know that x is *not* an element of $B\cap C$. So, this says that either $x\notin B$ or $x\notin C$. There are two cases. If $x\notin B$ then $x\in\overline{B}$ and so $x\in A\cap\overline{B}$. So, I can say that $x\in(A\cap\overline{B})\cup D$ where D is any set in the world since we know $x\in A\cap\overline{B}$. Let $D=A\cap\overline{C}$. Then we have shown that $x\in(A\cap\overline{B})\cup(A\cap\overline{C})$. In the other case, suppose that $x\notin C$. Then we know that $x\in\overline{C}$ and so $x\in A\cap\overline{C}$. So, it follows again that $x\in E\cup(A\cap\overline{C})$ where E is any set in the world. So, let $E=A\cap\overline{B}$. Then we have shown (again) that $x\in(A\cap\overline{B})\cup(A\cap\overline{C})$. Since $x\in B\cap\overline{(A\cap C)}$ was arbitrary it follows that $A\cap\overline{(B\cap C)}\subseteq(A\cap\overline{B})\cup(A\cap\overline{C})$.

We still need to show (2) is true. So, suppose that $x\in(A\cap\overline{B})\cup(A\cap\overline{C})$. Then either $x\in A\cap\overline{B}$ or $x\in A\cap\overline{C}$. First let's suppose that $x\in A\cap\overline{B}$. Then we know that $x\in A$ and $x\notin B$. Note that if $x\notin B$ then $x\notin B\cap G$ where G is any set in the world. So, $x\in A$ and $x\notin B\cap G$. Since G is allowed to be any set in the world, let $G=C$. Then we have shown that $x\in A$ and $x\notin B\cap C$. Hence $x\in A\cap\overline{(B\cap C)}$. The other case is similar. Suppose that $x\in A\cap\overline{C}$. Then $x\in A$ and $x\notin C$. So, by a similar argument as before, $A\notin C$ implies $A\notin B\cap C$. So, $x\in A$ and $x\notin B\cap C$. Hence $x\in A\cap\overline{(B\cap C)}$. In either case we have shown that the arbitrary element $x\in(A\cap\overline{B})\cup(A\cap\overline{C})$ also had to live in $A\cap\overline{(B\cap C)}$. So, it follows that $\left(A\cap\overline{B}\right)\cup\left(A\cap\overline{C}\right)\subseteq A\cap\left(\overline{B\cap C}\right)$ as desired.

We have shown that both (1) and (2) are true. So, by definition of equality for sets, this shows that $A\cap\left(\overline{B\cap C}\right)=\left(A\cap\overline{B}\right)\cup\left(A\cap\overline{C}\right)$ and we have completed the proof. □

(6) (15 points)

Let $g : [0, \infty) \longrightarrow [3, \infty)$ be defined by

$$g(x) = 3x^2 + 3.$$

Prove that g is bijective. (Drawing a picture is not sufficient.)

Proof. We need to show that the function g is both injective and surjective. Let's begin with injectivity.

We need to show that for all $x_1, x_2 \in [0, \infty)$, if $g(x_1) = g(x_2)$, then $x_1 = x_2$. So, let x_1 and x_2 be arbitrary elements of $[0, \infty)$ and assume that $g(x_1) = g(x_2)$. Then we know that $3x_1^2 + 3 = 3x_2^2 + 3$. So,

$$\begin{aligned} 3x_1^2 + 3 &= 3x_2^2 + 3 \\ \Rightarrow 3x_1^2 &= 3x_2^2 \\ \Rightarrow x_1^2 &= x_2^2 \\ \Rightarrow x_1 &= \pm x_2 \end{aligned}$$

But, x_1 and x_2 are elements of $[0, \infty)$ so, $x_1 \geq 0$ and $x_2 \geq 0$. So, we can ignore the negative solution and deduce that $x_1 = x_2$ as desired.

Now we need to show that g is surjective. That is, we must show that every element $y \in [3, \infty)$ is the image of some element $x \in [0, \infty)$ by g. That is, given $y \in [3, \infty)$ we need to show that there exists an $x \in [0, \infty)$ so that $g(x) = y$. So, lets see if we can solve for it.

$$\begin{aligned} g(x) &= y \\ \Rightarrow 3x^2 + 3 &= y \\ \Rightarrow 3x^2 &= y - 3 \\ \Rightarrow x^2 &= \frac{y-3}{3} \\ \Rightarrow x &= \pm\sqrt{\frac{y-3}{3}} \end{aligned}$$

So, $x \in [0, \infty)$ provided the term under the radical is positive. We need only verify that we are not trying to take the square root of a negative number. Remember $y \in [3, \infty)$, so $y \geq 3$. Therefore, $y - 3 \geq 0$ and so the fraction $(y - 3)/3 \geq 0$ and we can take the square root. So, there is an element $x \in [0, \infty)$ so that $g(x) = y$ and, since y was arbitrary, it follows that g is surjective.

We have shown that the function g is both injective and surjective. So, we have shown that it is bijective and we have finished the proof. $\square$