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Test 3

Spring 2003

CS/MATH 2610

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Directions : You have 75 minutes to complete all 6 problems on this exam. There are a possible 100 points to be earned. You may not use your book or any notes. Please be sure to show all pertinent work. *An answer with no work will receive very little credit!* If any portion of the exam is unclear please come to me and I will elaborate provided I can do so without giving away the problem.

(1) (20 points)

(a) State the well ordering principle.

(b) State the principle of mathematical induction.

(c) State the binomial theorem.

(d) State the generalized pigeonhole principle.

(e) What are the formulas for $P(n, k)$ and $C(n, k)$ (where $n \geq k$)?

Solution :

(a) Every nonempty set of nonnegative integers has a least element.

(b) If S is any subset of nonnegative integers so that the following two conditions hold:

1) $1 \in S$.

2) If $n \in S$, then $n + 1 \in S$.

Then, $S = \mathbb{N}$.

(c) Let x and y be variables, and let n be a nonnegative integer. Then,

$$(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j.$$

(d) If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

(e)

$$P(n, k) = \frac{n!}{(n-k)!}$$

$$C(n, k) = \frac{n!}{k!(n-k)!}.$$

(2) (15 points)

Prove that $1 + 5 + 9 + \cdots + (4n - 3) = n\,(2n - 1)$ for all $n \in \mathbb{N}$.

Solution : We will proceed using the principle of mathematical induction. First, define

$$S = \{n \in \mathbb{N} | 1 + 5 + 9 + \cdots + (4n - 3) = n\,(2n - 1)\}\,.$$

Then, we only need to show that $S = \mathbb{N}$. To do this, note that the principle of mathematical induction asserts that it is sufficient to show two things. Specifically, we need to show

(Base Case) 1) $1 \in S$, and

(IHOP Case) 2) if $n \in S$, then $n + 1 \in S$.

So, we begin with the base case. To show that $1 \in S$, we need to show that $1 \in \mathbb{N}$ and that $1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1)$ holds when $n = 1$. Certainly $1 \in \mathbb{N}$ since $\mathbb{N} = \{1, 2, 3, \ldots\}$. Moreover, when $n = 1$ the equation in question becomes $1 = 1(2 \cdot 1 - 1)$, which is true. Therefore $1 \in S$.

Next we begin step two. Let us make the induction hypothesis (IHOP). Assume that $n \in S$, that is assume $1 + 5 + 9 + \cdots (4n - 3) = n(2n - 1)$. Then we wish to use this assumption to prove that $n + 1 \in S$, which is to say $1 + 5 + 9 + \cdots + (4n - 3) + (4(n + 1) - 3) = (n + 1)(2(n + 1) - 1)$. To do this we will show that the left side of the equation can be made to reduce to the right side of the equation with the (IHOP) assumption. Indeed,

$$\begin{aligned} LS &= 1 + 5 + 9 + \cdots + (4n - 3) + (4(n + 1) - 3) \\ &= [1 + 5 + 9 + \cdots + (4n - 3)] + (4(n + 1) - 3) \\ &= n(2n - 1) + (4(n + 1) - 3) \text{ (by (IHOP))} \\ &= 2n^2 - n + 4n + 4 - 3 \\ &= 2n^2 + 3n + 1. \end{aligned}$$

Now, lets see if we can get the right side to be equal to the left side.

$$\begin{aligned} RS &= (n + 1)(2(n + 1) - 1) \\ &= (n + 1)(2n + 2 - 1) \\ &= (n + 1)(2n + 1) \\ &= 2n^2 + 3n + 1. \end{aligned}$$

So, we have shown that the left side of the equation is equal to the right side, provided that we assume (IHOP). That is, we have proven that if $n \in S$, then $n + 1 \in S$. So, by the principle of mathematical induction we have shown $S = \mathbb{N}$ as desired.

(3) (15 points)

Let $A = \{1, 2, 3, \ldots, n\}$ and let $B = \{0, 1, 2\}$.

(a) How many functions $f : A \longrightarrow B$ exist?

(b) How many injective functions $g : A \longrightarrow B$ exist?

(c) How many functions $h : A \longrightarrow B$ exist provided $h(1) \neq 2$ and $h(n) \neq 0$?

Solution :

(i) There are exactly three choices for $f(1)$, for each of those initial choices there are three choices for $f(2)$, ..., for each of the previous choices of $f(1)$ through $f(n-1)$ there are three choices for $f(n)$. So, by the multiplication principle, there are 3^n possible functions. This question is no different than asking how many passwords of length n exist if there are only three characters allowed.

(ii) There are exactly 0 injective functions whenever $n \geq 4$. If $n = 3$, then there are three places to choose from for $g(1)$ which leaves two places to choose for $g(2)$ and then exactly one possibility for $g(3)$ for a total of $3 \cdot 2 \cdot 1 = 6$ possible injective functions when $n = 3$. If $n = 2$, then there are three choices for $g(1)$ and two choices for $g(2)$ for a total of 6 possible injective functions again. Finally, if $n = 1$, then there are three injective functions $g(1) = 0$, $g(1) = 1$, and $g(1) = 2$.

(iii) First notice that there are no restrictions on $h(2), h(3), \ldots, h(n-1)$. So, there are 3^{n-2} possible ways to fill in these values since there are $n - 2$ positions to fill and each position has three possible values 0, 1, or 2. Now, for each way we have to fill the center elements there are two choices for $h(1)$ (specifically $h(1) = 0$ and $h(1) = 1$) and two choices for $h(n)$ (specifically $h(n) = 1$ and $h(n) = 2$). So, using the multiplication principle again we see that there are a total of $2 \cdot 3^{n-2} \cdot 2$ such functions.

(4) (15 points)

Show that if there are 100,000,000 persons employed in the U.S. who earn less than \$1,000,000 in a given year, then there are at least two people that earned the same amount of money, to the penny, in that year.

Solution : If we have 100,000,000 folks and each one of them earned less than \$1,000,000, then we know that each person earned less than 100,000,000 pennies. So, with the assumption that everyone earned at least 1 penny, it follows that there must be two people that earned the same amount, to the penny, according to the pigeonhole principle. Indeed, let each person represent a distinct ball and let each bin be described by the number of pennies people earn. Then, we see that we have 100,000,000 balls that we are trying to stick in 99,999,999 (since everyone earned *less* than \$1,000,000 they earned at most \$999,999.99). We have one more ball than we have bins, so we have to double up somewhere. So, there is at least one bin (representing a specific amount of pennies) that holds two people. Those two people earned the same amount of money, to the penny, last year.

(5) (15 points)

What is the coefficient of the term $x^{28}y^{72}$ in the expansion of $(x+y)^{100}$? Justify your answer with a combinatorial proof.

Solution : The coefficient should be exactly

$$C(100,28) = \left(\begin{array}{c} 100 \\ 28 \end{array} \right) = \frac{100!}{28!(100-28)!}.$$

The fact that the coefficient is equal to the number of ways to choose 28 elements out of 100 can be see in the following way. If we were to write out the product

$$(x+y)^{100} = (x+y)(x+y)(\cdots)(x+y)$$

then we need to think of how many ways we can get terms that have the product $x^{28}y^{72}$. Each time we obtain one of these terms, it must have come from choosing x from exactly 28 of the $(x+y)$ terms above, of which there are 100. So, it follows that we need to know the number of ways we can select 28 of the $(x+y)$ terms to pull an x out of. This is exactly $C(100,28)$. Notice that we could have counted the y's instead to obtain the coefficient $C(100,72)$ which is no different than $C(100,28)$ according to the identity $C(n,k) = C(n,n-k)$.

(6) (15 points)

In how many ways can the symbols $\{A, B, C, D, E, F\}$ be arranged provided that we insist that the symbol A come before the symbol C?

Solution : (The slick way that some of you thought to solve the problem.) There are a total of $P(6,6) = 6!$ ways to organize the symbols $\{A, B, C, D, E, F\}$. Exactly half of these are configurations in which A comes before C (it either does or doesn't, not both). Therefore, the total number of ways to arrange these six symbols so that A appears before C is $6!/2 = 360$.

(The non-slick way that I used to solve the problem). Alternatively , there is another way to count the possible configurations. The letter A can appear in any one of 6 positions. If C must come after A, then we can describe the situation as follows. If A is in the first position, then C can be in any of the remaining 5 positions, lets denote this is (A, _ , _ , _ , _ , _). Then, the letter C can be in any of these positions and the remaining four can be filled up in $P(4,4)$ different ways. This gives us a total of $1 \cdot 5 \cdot P(4,4)$ ways to arrange the symbols when A appears in the first position.
If A appears in the second position, then we are in the scenario that looks like (X, A, _ , _ , _ , _), where the X denotes a position that C cannot be in. So, C can appear in any of the remaining 4 positions and the last 4 letters can be put down in any remaining position in a total of $P(4,4)$ ways again. So, the total number of configurations in which A appears in the second spot will be $1 \cdot 4 \cdot P(4,4)$.

Similarly, there are $1 \cdot 3 \cdot P(4,4)$ possible configurations of the form (X, X, A, _ , _ , _) (where A is in the third spot), there are $1 \cdot 2 \cdot P(4,4)$ possible configurations of the form (X, X, X, A, _ , _), there are $1 \cdot 1 \cdot P(4,4)$ possible configurations of the form (X, X, X, X, A, _), and finally, if A is in the last spot (X, X, X, X, X, A) then there is no place to put C so there are no configurations where A is the last element. Adding all of these cases up (which are distinct because A is in a different position in each case) we obtain the number,

$$\begin{aligned} &1 \cdot 5 \cdot P(4,4) + 1 \cdot 4 \cdot P(4,4) + 1 \cdot 3 \cdot P(4,4) \\ &\quad + 1 \cdot 2 \cdot P(4,4) + 1 \cdot 1 \cdot P(4,4) + 1 \cdot 0 \cdot P(4,4) \\ &= 5! + 4 \cdot 4! + 3 \cdot 4! + 2 \cdot 4! + 4! \\ &= (15)4! = 15 \cdot 24 = 360. \end{aligned}$$