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Test 1
Spring 2005
CSCI/MATH 2610
February 10, 2005

Directions : You have 75 minutes to complete all 6 problems on this exam. There are a possible 110 points to be earned with 10 extra credit points built in. You may not use your book or any notes. Please be sure to show all pertinent work. *An answer with no work will receive very little credit!* If any portion of the exam is unclear please come to me and I will elaborate provided I can do so without giving away the problem.

1. (25 points)
   Answer each of the following questions.

   (a) What is the definition of an injective function.

   Solution : A function $f : A \longrightarrow B$ is **injective** if and only if $f(x) = f(y)$ implies $x = y$.

   (b) What is the definition of a surjective function.

   Solution : A function $f : A \longrightarrow B$ is **surjective** if and only if for all $y \in B$ there exist an $x \in A$ so that $f(x) = y$.

   (c) What is the negation of the statement $\exists x \exists y (P(x, y) \vee Q(x, y))$?

   Solution : $\forall x \forall y (\sim P(x, y) \wedge \sim Q(x, y))$.

   (d) Let A and B be sets. Define $A \times B$.

   Solution : $A \times B = \{(a, b) | a \in A \wedge b \in B\}$.

   (e) Let $A = \{\emptyset, \{a\}, \heartsuit\}$ what is $\mathcal{P}(A)$? List the elements. ($\mathcal{P}(A)$ denotes the power set of A.)

   Solution :

   $$\mathcal{P}(A) = \{\emptyset, \{\emptyset\}, \{\{a\}\}, \{\heartsuit\}, \{\emptyset, \{a\}\}, \{\emptyset, \heartsuit\}, \{\{a\}, \heartsuit\}, \{\emptyset, \{a\}, \heartsuit\}\}.$$

2. (15 points)
Let P, Q, and R be propositions. Prove or disprove that

$$[((P \vee Q) \wedge \sim P) \to Q] \vee (R \to \sim R) \equiv \mathbf{T}.$$

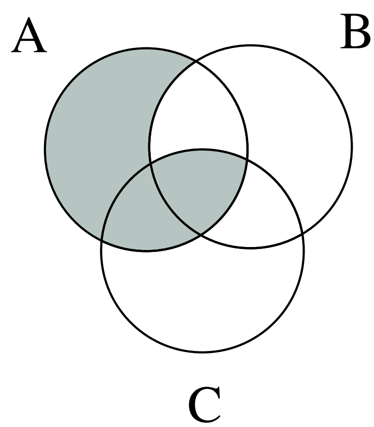
Solution : Notice first that if $[((P \vee Q) \wedge \sim P) \to Q]$ or $(R \to \sim R)$ is true, then we will have the desired result since they are joined by an OR. Indeed, $[((P \vee Q) \wedge \sim P) \to Q] \equiv \mathbf{T}$.

$$\begin{aligned}
[((P \vee Q) \wedge \sim P) \to Q] &\equiv \sim ((P \vee Q) \wedge \sim P) \vee Q \text{ (since } p \to q \equiv \sim p \vee q) \\
&\equiv (\sim (P \vee Q) \vee \sim (\sim P)) \vee Q \text{ (DeMorgan's law)} \\
&\equiv (\sim (P \vee Q) \vee P) \vee Q \text{ (Double negation)} \\
&\equiv ((\sim P \wedge \sim Q) \vee P) \vee Q \text{ (DeMorgan's law)} \\
&\equiv (P \vee (\sim P \wedge \sim Q)) \vee Q \text{ (Commutativity)} \\
&\equiv ((P \vee \sim P) \wedge (P \vee \sim Q)) \vee Q \text{ (Distributivity)} \\
&\equiv ((\mathbf{T} \wedge (P \vee \sim Q)) \vee Q \text{ (Negation law)} \\
&\equiv (P \vee \sim Q) \vee Q \text{ (Identity law)} \\
&\equiv P \vee (\sim Q \vee Q) \text{ (Associativity)} \\
&\equiv P \vee (\mathbf{T}) \text{ (Negation law)} \\
&\equiv \mathbf{T} \text{ (Domination law).}
\end{aligned}$$

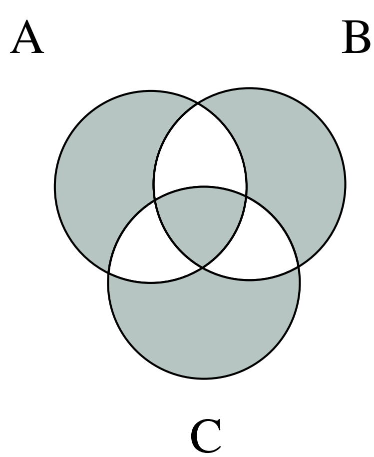
So, it follows that $[((P \vee Q) \wedge \sim P) \to Q] \vee (\text{Anything in the world}) \equiv \mathbf{T}$. Hence, $[((P \vee Q) \wedge \sim P) \to Q] \vee (R \to \sim R) \equiv \mathbf{T}$.

3. (18 points)
Use Venn diagrams to sketch the following sets.

(a) $A \cap \left(\overline{B - C}\right)$



(b) $[A \cup B \cup C) - ((A \cap B) \cup (A \cap C) \cup (B \cap C))] \cup (A \cap B \cap C)$



4. (15 points)
Let n be an integer. Prove that if $5n + 3$ is odd, then n is even.

Solution : If you try a direct proof, you will see that you get stuck pretty quickly. So, we can try proofs by contradiction or contrapositive. Each will work.

Contradiction: Assume $5n + 3$ is odd and n is odd. Then, $n = 2k + 1$ for some $k \in \mathbb{Z}$ and so $5n + 3 = 5(2k + 1) + 3 = 10k + 8 = 2(5k + 4)$ which is even. This is a contradiction since we assumed $5n + 3$ was odd. Therefore, it follows that if $5n + 3$ is odd, then n is even.

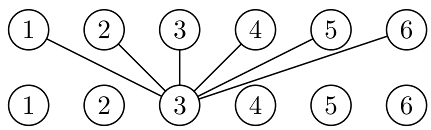
Contrapositive: We will prove the logically equivalent statement "if n is odd, then $5n + 3$ is even." Assume n is odd. Then $n = 2k + 1$ for some $k \in \mathbb{Z}$. Therefore, $5n + 3 = 5(2k + 1) + 3 = 10k + 8 = 2(5k + 4)$ which is even as desired.

5. (16 points)
Define a function $f : \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$ that is

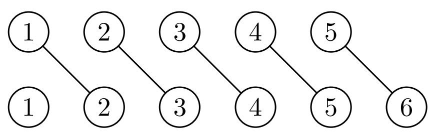
(a) neither injective or surjective.
(b) injective but not surjective.
(c) surjective but not injective.
(d) a bijection other than the identity function $f(x) = x$.

Solution :

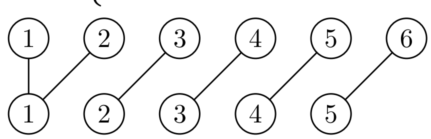
(a) $f(x) = 3$



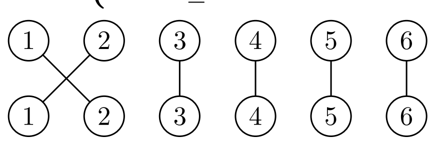
(b) $f(x) = x + 1$



(c) $f(x) = \begin{cases} 1 & x \leq 2 \\ x - 1 & x > 2 \end{cases}$



(d) $f(x) = \begin{cases} 2 & x = 1 \\ 1 & x = 2 \\ x & x \geq 3 \end{cases}$

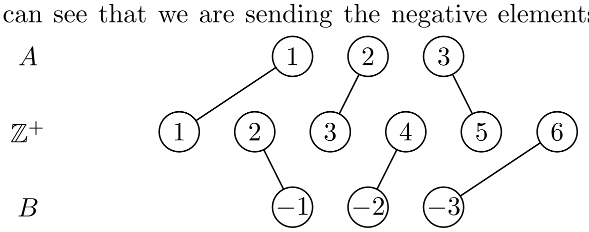


6. (20 points)
Let $A = \{1, 2, 3, \ldots\}$ and $B = \{-1, -2, -3, \ldots\}$. Prove $|A \cup B| = |\mathbb{Z}^+|$.

Solution : We must show that there exists a bijection $f : A \cup B \longrightarrow \mathbb{Z}^+$. I claim that the following function is such an animal.

$$f(x) = \begin{cases} 2x - 1 & x \in A \\ -2x & x \in B \end{cases}$$

So you can see that we are sending the negative elements (elements of



B) to the even guys and the positive elements (elements of A) to the odd guys. To show that this is a bijection we must show that it is injective as well as surjective.

(Injectivity) We must show that if $f(x) = f(y)$, then $x = y$. So, assume that $f(x) = f(y)$. Since $f(x) = f(y) \in \mathbb{Z}^+$ we know that it is either even or odd. Assume first that it is even. That is, assume $f(x) = 2k = f(y)$ for some $k \in \mathbb{Z}$. The only elements mapped to even integers are elements from B. So, we know that $x = -i$ for some positive integer i and $y = -j$ for some positive integer j. But $f(x) = f(y)$ implies $f(-i) = f(-j)$. But $f(-i) = 2i$ and $f(-j) = 2j$. If $f(x) = f(y)$, then $2i = 2j$ and thuse $i = j$. It follows that $x = y$.

Now, suppose that $f(x) = f(y)$ is an *odd* integer. Then $f(x) = 2k + 1 = f(y)$ for some $k \in \mathbb{Z}$. Therefore, x and y must be elements of A since they are the only elements mapped to odd integers. It follows that $x = i$ for some $i \in \mathbb{Z}^+$ and $y = j$ for some $j \in \mathbb{Z}^+$. Thus, $f(x) = f(y)$ implies $f(i) = f(j)$. By the definition of $f(x)$ this tells us, $2i - 1 = 2j - 1$. So, $i = j$ and again we have shown that $x = y$. This completes the proof that f is injective.

(Surjectivity) We must show that if $y \in \mathbb{Z}^+$, then there exists an $x \in \mathbb{Z}^+$ so that $f(x) = y$. There are again two cases. If $y = 2k$ for some $k \in \mathbb{Z}$, then let $x = -k \in B$. Then $f(x) = f(-k) = 2k = y$ as desired. Lastly, if $y = 2k + 1$ for some $k \in \mathbb{Z}$, then if we let $x = k + 1$ we see that $f(x) = f(k + 1) = 2(k + 1) - 1 = 2k + 1 = y$. This completes the proof that f is surjective. Since $f : A \cup B \longrightarrow \mathbb{Z}^+$ is injective and surjective we may finally deduce that $|A \cup B| = |\mathbb{Z}^+|$.