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Test 2

Spring 2005

CS/MATH 2610

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Directions : You have 75 minutes to complete all 7 problems on this exam. There are a possible 100 points to be earned. You may not use your book or any notes. Please be sure to show all pertinent work. *An answer with no work will receive very little credit!* If any portion of the exam is unclear please come to me and I will elaborate provided I can do so without giving away the problem.

(1) (20 points)
Let $a, b, m \in \mathbb{Z}$ with $m > 0$. Answer each of the following questions.

(a) Define what it means to say $f(x)$ is $O(g(x))$.

Solution : $f(x)$ is $O(g(x))$ provided there exist constants C and k so that $|f(x)| \leq C|g(x)|$ whenever $x > k$.

(b) What is the decimal expansion of the base b number $a_k a_{k-1} \ldots a_1 a_0$?

Solution : $a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b^1 + a_0 b^0$

(c) Define the statement $a \mid b$.

Solution : $a \mid b$ provided there exists and integer k so that $b = ak$.

(d) What does it mean to say the integers $a_1, a_2, \ldots, a_n$ are pairwise prime?

Solution : The integers $a_1, a_2, \ldots, a_n$ are pairwise prime provided $\gcd(a_i, a_j) = 1$ whenever $i \neq j$.

(e) Define the statement $a \equiv b$ (mod m).

Solution : $a \equiv b$ (mod m) provided there exists an integer q so that $a = qm + b$.

(2) (10 points)

Prove that $f(x)$ is $O(x)$ where

$$f(x) = \frac{2x^3 + 2x^2 + 2}{x^2 + x + 1}.$$

Solution : We are to find constants C and k so that $|f(x)| \leq C|x|$ whenever $x > k$. The witnesses $C = 2$ and $k = 1$ will work. Indeed,

$$\begin{aligned}
2|x| &= \left|2x\left(\frac{x^2+x+1}{x^2+x+1}\right)\right| \\
&= \left|\frac{2x^3+2x^2+2x}{x^2+x+1}\right| \\
&\geq \left|\frac{2x^3+2x^2+2}{x^2+x+1}\right| \text{ (since } 2x \geq 2 \text{ when } x > 1\text{)} \\
&= |f(x)|.
\end{aligned}$$

(3) (9 points)

Find the following products in the indicated bases without converting into decimal. You can, of course, convert into decimal to check your answer, but I would like to see the work done in the indicated bases.

(a) (Base 16)

$$\begin{array}{r} 6\mathrm{AF} \\ \underline{\times \mathrm{AF}} \end{array}$$

Solution :

$$491A1$$

(b) (Base 3)

$$\begin{array}{r} 2110 \\ \underline{\times 102} \end{array}$$

Solution :

$$222220$$

(c) (Base 10)

$$\begin{array}{r} 1024 \\ \underline{\times 512} \end{array}$$

Solution :

$$524288$$

(4) (15 points)

Use the Euclidean Algorithm to compute gcd$(159, 216)$. (You will receive very few points if you do not use the Euclidean Algorithm).

Solution :

$$\begin{aligned}
216 &= 159(1) + 57\\
159 &= 57(2) + 45\\
57 &= 45(1) + 12\\
45 &= 12(3) + 9\\
12 &= 9(1) + 3\\
9 &= 3(3) + 0.
\end{aligned}$$

Since 3 was the last nonzero remainder, it is the greatest common divisor.

(5) (15 points)

Prove that if $a, b, c \in \mathbb{Z}$ so that $a \mid b$ and $a \mid (b + c)$, then $a \mid c$.

Solution : We will prove this directly. Assume that $a, b, c \in \mathbb{Z}$ so that $a \mid b$ and $a \mid (b+c)$. Then, by the definition of divide we know that there exist constants m and n so that $b = an$ and $(b + c) = am$. Substituting the first equation into the second shows us $(an + c) = am$. Solving this equation for c completes the proof. Indeed, $c = am - an = a(m - n)$. Since $m - n \in \mathbb{Z}$ it follows immediately that $a \mid c$.

(6) (15 points)

Is there an integer x that leaves a remainder of 2 when divided by 4, leaves a remainder of 3 when divided by 5, and also leaves a remainder of 5 when divided by 9? If so why? (You can cite a theorem.) If there is more than one such number, what are they?

Solution : The Chinese Remainder theorem guarantees a solution since 4, 5, and 9 are pairwise relatively prime. The solution will be

$$\begin{aligned} x &= 2(5)(9)(\overline{5}_4)(\overline{9}_4) + 3(4)(9)(\overline{4}_5)(\overline{9}_5) + 5(4)(5)(\overline{4}_9)(\overline{5}_9) \\ &= 2(5)(9)(1)(1) + 3(4)(9)(4)(4) + 5(4)(5)(7)(2) \\ &= 90 + 1728 + 1400 \\ &= 3218. \end{aligned}$$

This is just one of the solutions. There are infinitely many since the solutions are given modulo $4 \cdot 5 \cdot 9$. So, all solutions are of the form

$$3218 + 180k$$

where k is any integer.

(7) (15 points)

Use mathematical induction to prove this formula for the sum of a finite number of terms of a geometric progression:

$$a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r - 1}, \text{ where } r \neq 1.$$

Solution : Let $P(n)$ be the proposition

$$P(n) \equiv \left(a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r - 1}, \text{ where } r \neq 1 \right)$$

First we will establish the base case $P(0)$. When $n = 0$ we have

$$\frac{ar^{0+1} - a}{r - 1} = \frac{ar - a}{r - 1} = a\frac{r - 1}{r - 1} = a$$

as it should. This finishes the base case.

Now, assume $P(n) \equiv T$ and we will use this to try to deduce that $P(n+1)$ is also true. Using the fact that $P(n)$ is true we see that

$$\begin{aligned} a + ar + ar^2 + \cdots + ar^n + ar^{n+1} &= \frac{ar^{n+1} - a}{r - 1} + ar^{n+1} \\ &= \frac{ar^{n+1} - a}{r - 1} + \frac{ar^{n+1}(r - 1)}{r - 1} \\ &= \frac{ar^{n+1} - a + ar^{n+2} - ar^{n+1}}{r - 1} \\ &= \frac{ar^{(n+1)+1} - a}{r - 1}. \end{aligned}$$

This finishes the induction since we have shown $P(n+1)$ to be true.