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Test 3

Spring 2005

CS/MATH 2610

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Directions : You have 75 minutes to complete all 7 problems on this exam. There are a possible 100 points to be earned. You may not use your book or any notes. Please be sure to show all pertinent work. *An answer with no work will receive very little credit!* If any portion of the exam is unclear please come to me and I will elaborate provided I can do so without giving away the problem.

(1) (20 points)
Let $k, n, r \in \mathbb{Z}$ with $n > k$. Answer each of the following questions.

(a) State the Product Rule.

Solution : Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and n_2 ways to do the second task after the first task has been done, then there are $n_1 n_2$ ways to do the procedure.

(b) State the Pigeonhole Principle.

Solution : If $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

(c) State the Binomial Theorem.

Solution : Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

(d) Define an r-combination.

Solution : An r-combination of elements of a set is an unordered selection of r elements from the set.

(e) Why is $\binom{n}{k} = \binom{n}{n-k}$?

Solution : If we pull k elements out of a bag that contained n elements, then there are exactly $n-k$ elements left in the bag. So, we can count k element subsets of a set with n elements while simultaneously counting the number of $n-k$ element subsets of a set with n elements.

(2) (10 points)

You, Wil Wheaton, Brent Spiner, and Michael Dorn are playing a game of cards using a standard 52 card deck. What is the total number of ways to deal five cards to all four of you? That is, how many different ways can we distribute twenty cards to four people where each person gets five cards.

Solution : There are a couple of solutions to this problem that I saw that were correct. The first is the one from the book. There are 52 cards in the deck and the first person receives 5 of those cards in one of $\binom{52}{5}$ ways. Then, the second person gets a different set of 5 cards from the remaining 47 cards in one of $\binom{47}{5}$ different ways. Continuing in this fashion, and using the product rule, we see that number of possible deals is

$$\binom{52}{5}\binom{47}{5}\binom{42}{5}\binom{37}{5} = 1,478,262,843,475,644,020,034,240.$$

A second solution is to first determine the number of ways we can obtain the 20 cards that will be distributed to the players. There are $\binom{52}{20}$ different ways to select these 20 cards. Then, the first person is dealt 5 of these 20 cards in one of $\binom{20}{5}$ different ways. Then, the second person gets 5 of the remaining 15 cards in one of $\binom{15}{5}$ different ways. Combining these data using the product rule, we see that the total number of ways to deal out the cards is

$$\binom{52}{20}\binom{20}{5}\binom{15}{5}\binom{10}{5}\binom{5}{5} = 1,478,262,843,475,644,020,034,240.$$

Neat! We have just given a combinatorial proof of the identity,

$$\binom{52}{5}\binom{47}{5}\binom{42}{5}\binom{37}{5} = \binom{52}{20}\binom{20}{5}\binom{15}{5}\binom{10}{5}\binom{5}{5}.$$

Huzzah!

(3) (9 points)

How many one-to-one functions are there from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ into the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Solution : We need only describe what the function does to each element in the domain $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. We can send the valie 1 to any of the ten values in the co-domain. Once we have made this choice, our only restriction on where we send the domain value of 2 is that we not send it to the same place. So, that leaves any 1 of 9 choies remaining. We an send the domain value of 3 to any one of the now eight remaining values in the o-domain. Continuing this reasoning and using the product rule we see that the total number of injective functions is

$$10 * 9 * 8 * 7 * 6 * 5 * 4 * 3 * 2 = 3,628,800.$$

(4) (15 points)

Suppose x_1, x_2, x_3, and x_4 are non-negative integers. How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 9.$$

Solution : This is a Lincoln Log$^{\text{TM}}$ problem! Imagine that we have 4 bins, one for x_1, one for x_2, one for x_3, and one for x_4. We now need to define where the borders of these bins are. This is where we use the Lincoln Logs$^{\text{TM}}$. We have 9 stars laid out in a row. If we place the three Lincoln Logs$^{\text{TM}}$ down then this will define where the bins are. For example, below the placement of the Lincoln Logs$^{\text{TM}}$ gives us the solution x_1 = 2, $x_2 = 1$, $x_3 = 4$, and $x_4 = 2$.

$$* * | * | * * * * | * *$$

So, we have reduced the problem to determining how many ways we can put down the Lincoln Logs$^{\text{TM}}$. We have a set of 12 positions total to fill, 9 stars and 3 Lincoln Logs$^{\text{TM}}$. We need to select 3 of these 12 positions for the Lincoln Logs$^{\text{TM}}$. This can be done in one of $\binom{12}{3}$ different ways. Therefore, the total number of solutions to the above equation is $\binom{12}{3} = 220$.

(5) (15 points)

Show that among any group of twenty six (not necessarily consecutive) integers, there are two with the same remainder when divided by 25.

Solution : When dividing by 25, the possible remainders are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, or 24. There are only 25 possible remainders. So, if we have a group of 26 numbers and we divide each of them by 25 we will have a set of 26 remainders. By the Pigeonhole principle, at least two of these remainders must be equal.

(6) (15 points)

Prove that if n and k are positive integers with $n \geq k$, then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Solution : This is Pascal's Identity. Since I didn't require the proof to be done combinatorially (it's in the book on page 320) I will do the proof algebraically.

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-(k-1))!} + \frac{n!}{k!(n-k)!} \\
&= \frac{n!k}{k!(n-(k-1))!} + \frac{n!(n-(k-1))}{k!(n-(k-1))!} \\
&= \frac{n!(k+(n-(k-1))}{k!(n-(k-1))!} \\
&= \frac{n!(n+1)}{k!((n+1)-k)!} \\
&= \frac{(n+1)!}{k!((n+1)-k)!} \\
&= \binom{n+1}{k}.
\end{aligned}$$

(7) (15 points)

Use a combinatorial proof to show that if n is a nonnegative integer, then

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2 . \text{ [Hint: (1)(e).]}$$

Solution : Using the hint I will rewrite the claim as follows

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} .$$

Once we have written the claim in this form it becomes more apparent that this is a special case of Vandermonde's Identity. So, the combinatorial proof will be similar (identical) to the proof of Vandermonde's Identity.

Suppose we had a bag containing 2n elements. Suppose we divided this bag into two smaller bags A and B which each contain n elements. On the left hand side of the claim we see that this number is the number of n elements subsets from the bag with $2n$ elements. So, how can we count the same thing and end up with the expression on the right. Well, suppose we have an arbitrary n element subset of the bag with $2n$ elements. It either has an element from A in it or it doesn't. So, we can count the number of n element subsets contained in the bag with $2n$ elements by counting the number of n element subsets that have 0 elements from A, 1 element from A, 2 elements from A, $\ldots$, $n-1$ elements from A, or all n elements from A and adding them up. Since the cases cannot occur simultaneously (we can't have an n element subset that contains exactly 3 elements from A while containing exactly 7 elements from A) we see that we do not need to worry about inclusion-exclusion. Now, how many ways can we have an n element subset that has no elements in A? Well, that means that I have chosen 0 elements from A, a set containing n elements. I can do this $\binom{n}{0}$ different ways. For each of these choices, I can select the n remaining elements of the set from B, a set with n elements, in $\binom{n}{n}$ different ways. So, the total number of n element subsets from the bag that contains $2n$ elements which has no elements from A is $\binom{n}{0}\binom{n}{n}$.
Now, how many n element subsets have one element from A and $n-1$ elements from B. Well, there is $\binom{n}{1}$ different ways to choose the ne element from A. For each of these choices, there is $\binom{n}{n-1}$ different ways to choose the remaining $n-1$ elements from B. This gives us $\binom{n}{1}\binom{n}{n-1}$ different n element subsets that contain exactly one element from A.
Continuing to the general case, we want to know how many ways we can get an n element subset that contains k elements from A and $n-k$ elements from B. This number is exactly $\binom{n}{k}\binom{n}{n-k}$. Adding all of these cases up we complete the

proof

$$\binom{2n}{n} = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + cdots \binom{n}{k}\binom{n}{n-k} + cdots + \binom{n}{n}\binom{n}{0}$$

$$= \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k}^2 .$$