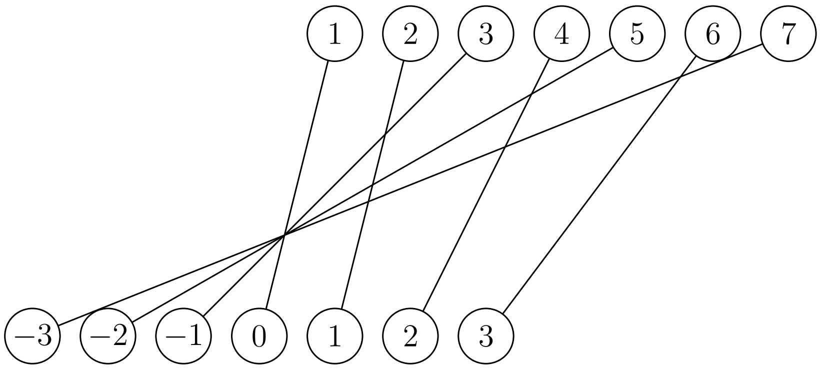
MATH 2610
Discrete Mathematics for Computer Science
January 26, 2005
Addendum

Today I gave a heuristic proof stating the the cardinality of $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ was the same as the cardinality of $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. I would now like to justify that claim with a more rigorous proof. This will also serve as a good way to provide an example on proofs regarding injectivity and surjectivity.

Claim 1. $|\mathbb{Z}^+| = |\mathbb{Z}|$.

Proof. To show that the cardinality of two sets A and B are equal, it suffices to show that there exists a bijection $f : A \longrightarrow B$. Recall that a bijection is a function that is both injective (one-to-one) and surjective (onto). So, our goal for this proof will be to find a bijection $f : \mathbb{Z}^+ \longrightarrow \mathbb{Z}$. The idea is just like we had in class. We can draw a picture to try to find out what a possible function may be.



Now let see if we can devise a closed form for this function and see if we can prove that it is bijective. First, notice that all of the even numbers in the top row are paired with positive numbers on the bottom row while odd numbers are paired with negative numbers and zero. First, lets focus on the even guys. We see that $2 \mapsto 1$, $4 \mapsto 2$, $6 \mapsto 3$, etc... So, it looks like any even number $2k$ will get sent to k. Let's look at the odd guys now. We see that $1 \mapsto 0$, $3 \mapsto -1$, $5 \mapsto -2$, $7 \mapsto -3$ etc... An arbitrary odd number looks like $2k + 1$ and it looks like we want to send this odd number to $-k$. So, it looks like the function we want is

$$f(n) = \begin{cases} k & \text{if } n = 2k \text{ for some } k \\ -k & \text{if } n = 2k+1 \text{ for some } k \end{cases}$$

You can check and see that $f(1) = f(2(0) + 1) = -0 = 0$, $f(2) = f(2(1)) = 1$, $f(3) = f(2(1) + 1) = -1$. Indeed, this function agrees with our picture above! Let's see if we can prove that it is a bijection. First we will show that it is injective (one-to-one). To see this first we must recall the definition of one-to-one. A function f is one-to-one if and only if whenever $f(x) = f(y)$ it must also be true that $x = y$. So, we will assume that we have some counting numbers x and y in $\mathbb{Z}^+$ so that $f(x) = f(y)$. Well, if $f(x) = f(y)$ then either $f(x) = f(y) < 0$, $f(x) = f(y) = 0$, or $f(x) = f(y) > 0$. So, we need to show that in any of these three possible cases it must be true that $x = y$.

Suppose first that $f(x) = f(y) < 0$. Then we know that x and y must both be odd numbers since only odd numbers are sent to negative numbers. Therefore, $x = 2k + 1$ for some k and $y = 2\ell + 1$ for some ℓ. We know that $f(x) = f(y)$, and we can rewrite this as $f(2k+1) = f(2\ell+1)$. By the definition of f we can extend this statement to say $-k = f(2k+1) = f(2\ell+1) = -\ell$, which tells us $k = \ell$. But if $k = \ell$, then $2k + 1 = 2\ell + 1$ and so $x = y$.

The exact same argument works for the case $f(x) = f(y) = 0$ since only an odd number can be sent to zero (the first even number $2 \in \mathbb{Z}^+$ gets sent to $+1$ and every other even number gets sent to larger values). Indeed, if $f(x) = f(y) = 0$, then $x = y = 1$.

Finally suppose that $f(x) = f(y) > 0$. Then we know that x and y must be even. So $x = 2k$ for some k and $y = 2\ell$ for some ℓ. Rewriting our assumption we have $f(2k) = f(2\ell)$ which implies $k = f(2k) = f(2\ell) = \ell$, and again we see that this must mean that $x = y$. We can now conclude that f is one-to-one.

It remains only to show that f is surjective. To do this recall that a function $f : A \longrightarrow B$ is surjective if and only if for each $b \in B$ there is an element $a \in A$ so that $f(a) = b$. That is, in our case, we need to make sure that every element of $\mathbb{Z}$ gets hit by some element in $\mathbb{Z}^+$ by f. So, suppose that y is any element in $\mathbb{Z}$. We need to cook up an element x of $\mathbb{Z}^+$ so that $f(x) = y$. Again, there are three cases.

If $y < 0$, then $y = (-1)k$ for some positive integer k. Then let $x = 2k + 1$. By definition of f we see that $f(x) = f(2k + 1) = -k = y$.

If $y =$, then let $x = 1$, since $f(1) = 0$.

If $y > 0$, then $y = \ell$ for some positive integer ℓ. Then let $x = 2\ell$. By definition of f we see that $f(x) = f(2\ell) = \ell = y$. This complets the proof of the assertion that f is surjective. Since we had shown previously that f was injective, we can now say that f is bijective. Thus $|\mathbb{Z}^+| = |\mathbb{Z}|$. $\square$