MATH 2610
Discrete Mathematics for Computer Science
March 24, 2005
Addendum

Let's see if I can give a better explanation of the binomial theorem.

Theorem 1. *Let x and y be variables, and let n be a nonnegative integer. Then,*

$$\begin{aligned}(x+y)^n &= \sum_{j=0}^{n} \binom{n}{j} x^{n-j}y^j \\ &= \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1}y^1 + \cdots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} x^0 y^n.\end{aligned}$$

Now, let's look at a specific example and see if we can construct the general proof from that. To do this, I'll consider $(x+y)^5$. First, write out $(x+y)^5$ as

$$(x+y)^5 = (x+y)(x+y)(x+y)(x+y)(x+y).$$

Lets start with the term x^5y^0. This term *must* have come from multiplying together all five of the $(x+y)$'s. There is only one way to do this (since there is only one group of the $(x+y)$'s that contains all of the $(x+y)$'s). So, the coefficient of the term x^5y^0 must be a 1. Now, how can we get a term that includes x^4y^1? well, we must have multiplied together *four* of the $(x+y)$'s in order to get the exponent of the x term up to a four. How many ways can we do this? Well let's write them down. I'll denoted which of four of the five $(x+y)$'s I am using with bold font. There are five possibilities. Specifically they are

$$\begin{gathered}(x+y)(\mathbf{x}+\mathbf{y})(\mathbf{x}+\mathbf{y})(\mathbf{x}+\mathbf{y})(\mathbf{x}+\mathbf{y})\\(\mathbf{x}+\mathbf{y})(x+y)(\mathbf{x}+\mathbf{y})(\mathbf{x}+\mathbf{y})(\mathbf{x}+\mathbf{y})\\(\mathbf{x}+\mathbf{y})(\mathbf{x}+\mathbf{y})(x+y)(\mathbf{x}+\mathbf{y})(\mathbf{x}+\mathbf{y})\\(\mathbf{x}+\mathbf{y})(\mathbf{x}+\mathbf{y})(\mathbf{x}+\mathbf{y})(x+y)(\mathbf{x}+\mathbf{y})\\(\mathbf{x}+\mathbf{y})(\mathbf{x}+\mathbf{y})(\mathbf{x}+\mathbf{y})(\mathbf{x}+\mathbf{y})(x+y).\end{gathered}$$

Now, you might be thing "Whoa hoss! What if I want to count the number of terms of the form x^4y^1 by dealing with y's instead of x's! How do I know this trick will work if I ask how many ways are there that I can multiply together just *one* of the $(x+y)$'s since the power of the y term is a one?" I would say, well we can write out the same list as above and just look at it's photo negative. Again, I will denote in bold font the terms that can contribute to the term x^4y^1 by counting the ways the $(x+y)$'s can be multiplied together to give me a power of y^1.

$$\begin{gathered}(\mathbf{x}+\mathbf{y})(x+y)(x+y)(x+y)(x+y)\\(x+y)(\mathbf{x}+\mathbf{y})(x+y)(x+y)(x+y)\\(x+y)(x+y)(\mathbf{x}+\mathbf{y})(x+y)(x+y)\\(x+y)(x+y)(x+y)(\mathbf{x}+\mathbf{y})(x+y)\\(x+y)(x+y)(x+y)(x+y)(\mathbf{x}+\mathbf{y}).\end{gathered}$$

So, either way you count it there must be five terms of the form x^4y^1. Next let's concentrate on how many terms there can be of the form x^3y^2. We can do this one of two ways (and I will show that they each give the same answer). We can notice that the power of x is three and so we must have multiplied together three of the $(x+y)$'s, or we can realize that the power of the y is two and count the ways we can multiply two of the $(x+y)$'s. First, let's write down all the ways we can multiply together three of the $(x+y)$'s.

$$
\begin{gathered}
(\mathbf{x}+\mathbf{y})(\mathbf{x}+\mathbf{y})(\mathbf{x}+\mathbf{y})(x+y)(x+y) \\
(\mathbf{x}+\mathbf{y})(\mathbf{x}+\mathbf{y})(x+y)(\mathbf{x}+\mathbf{y})(x+y) \\
(\mathbf{x}+\mathbf{y})(\mathbf{x}+\mathbf{y})(x+y)(x+y)(\mathbf{x}+\mathbf{y}) \\
(\mathbf{x}+\mathbf{y})(x+y)(\mathbf{x}+\mathbf{y})(\mathbf{x}+\mathbf{y})(x+y) \\
(\mathbf{x}+\mathbf{y})(x+y)(\mathbf{x}+\mathbf{y})(x+y)(\mathbf{x}+\mathbf{y}) \\
(\mathbf{x}+\mathbf{y})(x+y)(x+y)(\mathbf{x}+\mathbf{y})(\mathbf{x}+\mathbf{y}) \\
(x+y)(\mathbf{x}+\mathbf{y})(\mathbf{x}+\mathbf{y})(\mathbf{x}+\mathbf{y})(x+y) \\
(x+y)(\mathbf{x}+\mathbf{y})(\mathbf{x}+\mathbf{y})(x+y)(\mathbf{x}+\mathbf{y}) \\
(x+y)(\mathbf{x}+\mathbf{y})(x+y)(\mathbf{x}+\mathbf{y})(\mathbf{x}+\mathbf{y}) \\
(x+y)(x+y)(\mathbf{x}+\mathbf{y})(\mathbf{x}+\mathbf{y})(\mathbf{x}+\mathbf{y}).
\end{gathered}
$$

Now let's write down the number of ways we can multiply together two of the $(x+y)$'s and see that we get the same thing. Again, notice that if we have pulled out three terms before, that lea's us with two. So, our list will again look like a photo negative of the last list.

$$
\begin{gathered}
(x+y)(x+y)(x+y)(\mathbf{x}+\mathbf{y})(\mathbf{x}+\mathbf{y}) \\
(x+y)(x+y)(\mathbf{x}+\mathbf{y})(x+y)(\mathbf{x}+\mathbf{y}) \\
(x+y)(x+y)(\mathbf{x}+\mathbf{y})(\mathbf{x}+\mathbf{y})(x+y) \\
(x+y)(\mathbf{x}+\mathbf{y})(x+y)(x+y)(\mathbf{x}+\mathbf{y}) \\
(x+y)(\mathbf{x}+\mathbf{y})(x+y)(\mathbf{x}+\mathbf{y})(x+y) \\
(x+y)(\mathbf{x}+\mathbf{y})(\mathbf{x}+\mathbf{y})(x+y)(x+y) \\
(\mathbf{x}+\mathbf{y})(x+y)(x+y)(x+y)(\mathbf{x}+\mathbf{y}) \\
(\mathbf{x}+\mathbf{y})(x+y)(x+y)(\mathbf{x}+\mathbf{y})(x+y) \\
(\mathbf{x}+\mathbf{y})(x+y)(\mathbf{x}+\mathbf{y})(x+y)(x+y) \\
(\mathbf{x}+\mathbf{y})(\mathbf{x}+\mathbf{y})(x+y)(x+y)(x+y).
\end{gathered}
$$

In either case we count ten. So the coefficient of the term x^3y^2 must be a ten. But now we are done. By the symmetry illustrated by the photo negative idea, we see that there must also be ten terms of the form x^2y^3, five terms of the form x^1y^4, and one term of the form x^0y^5. So, we have shown (and you should verify) that

$$(x+y)^5 = 1x^5y^0 + 5x^4y^1 + 10x^3y^2 + 10x^2y^3 + 5x^1y^4 + 1x^{-}y^5.$$

The proof is on the next page.

Proof. We know that we can write

$$(x+y)^n = c_0x^ny^0 + c_1x^{n-1}y^1 + c_2x^{n-2}y^2 + \cdots + c_jx^{n-j}y^j + \cdots + c_nx^0y^n.$$

Therefore, it suffices to find all of the coefficients c_j for $j = 0, 1, \ldots, n$. Consider the general term $c_jx^{n-j}y^j$. This term is the result of multiplying together j of the $(x+y)$ terms in the expression $(x+y)^n$ since we see that there is a y^j in $c_jx^{n-j}y^j$. So, if we have a bag containing n terms that look like $(x+y)$, then this question is asking, how many different ways can we pull out j elements from the bag? This is exactly the number of subsets containing j elements that can be pulled from a set containing n elements. By definition, this is $\left(\begin{array}{c} n \\ j \end{array}\right)$. So, $c_j = \left(\begin{array}{c} n \\ j \end{array}\right)$ for each $j = 1, 2, \ldots, n$. This competes the proof since we have shown

$$(x+y)^n = \left(\begin{array}{c} n \\ 0 \end{array}\right) x^ny^0 + \left(\begin{array}{c} n \\ 1 \end{array}\right) x^{n-1}y^1 + \left(\begin{array}{c} n \\ 2 \end{array}\right) x^{n-2}y^2 + \cdots + \left(\begin{array}{c} n \\ j \end{array}\right) x^{n-j}y^j + \cdots + \left(\begin{array}{c} n \\ n \end{array}\right) x^0y^n$$

$$= \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array}\right) x^{n-j}y^j.$$

$\square$