Name: Dr. Mullikin

Test 1
Fall 2006
MATH 111 Section 02
September 13, 2006

Directions : You have 50 minutes to complete all 6 problems on this exam. There are a possible 100 points to be earned. You may not use your book, notes, or any graphing/programmable calculator. Please be sure to show all pertinent work. *An incorrect answer with no work will receive no credit!* If any portion of the exam is unclear please come to me and I will elaborate provided I can do so without giving away the problem.

1. (10 points)
Solve the following inequality, and express the solutions in terms of intervals if possible.

$$\frac{3}{x+3} \leq \frac{2}{x-5}$$

Solution : Don't forget, if there is an inequality invoved, we cannot cross multiply. Instead, move everything to one side using subtraction.

$$\frac{3}{x+3} \leq \frac{2}{x-5}$$

$$\frac{3}{x+3} - \frac{2}{x-5} \leq 0$$

$$\frac{3(x-5)}{(x+3)(x-5)} - \frac{2(x+3)}{(x+3)(x-5)} \leq 0$$

$$\frac{3x-15-2x-6}{(x+3)(x-5)} \leq 0$$

$$\frac{x-21}{(x+3)(x-5)} \leq 0.$$

The only points at which this expression can change from positive to negative or vice versa is when $x = 21$, $x = -3$, and $x = 5$. So, it suffices to find the value of $(x-21)/(x+3)(x-5)$ for some value of x smaller than -3, a value in between -3 and 5, a value in between 5 and 21, and a value larger than 21. Once this has been done we will know exactly where the expression $(x-21)/(x+3)(x-5)$ is positive and where it is negative. We begin with checking a number smaller than -3, say -4.

$$\frac{-4-21}{(-4+3)(-4-5)} = \frac{-25}{(-1)(-9)} = -\frac{25}{9} < 0$$

Since the outcome is negative, we know that $(x-21)/(x+3)(x-5) < 0$ for *all* $x \in (-\infty, -3)$. Next we check a number in between -3 and 5, say $x = 0$.

$$\frac{0-21}{(0+3)(0-5)} = \frac{-21}{(3)(-5)} = \frac{21}{15} > 0$$

Last we check the value at the points $x = 10$ and $x = 30$.

$$\frac{10-21}{(10+3)(10-5)} = \frac{-11}{(13)(5)} = -\frac{11}{65} < 0$$

$$\frac{30-21}{(30+3)(30-5)} = \frac{9}{(33)(25)} > 0$$

So, we have shown that the inequality is satisfied when x is in the set $(-\infty, -3) \cup (5, 21)$ and it remains to check if we include the endpoints. We cannot include the points $x = -3$ or $x = 5$ since we would be dividing by zero. However, we can include the point $x = 21$ since the fraction is zero there. So, our final solution is all x in the set $(-\infty, -3) \cup (5, 21]$.

2. (20 points)
   Prove that the points $A = (1, 1)$, $B = (2, 4)$, and $C = (8, 2)$ are the vertices of a right triangle.

   Solution : Pythagoras says that we need only show that the edges of the right triangle satisfy the pythagorean theorem. So, we compute the distances $d(A, B)$, $d(A, C)$, and $d(B, C)$.

$$d(A, B) = \sqrt{(2-1)^2 + (4-1)^2} = \sqrt{10}$$
$$d(A, C) = \sqrt{(8-1)^2 + (2-1)^2} = \sqrt{50}$$
$$d(B, C) = \sqrt{(8-2)^2 + (2-4)^2} = \sqrt{40}.$$

   Sure enough, $d(A, B)^2 + d(B, C)^2 = d(A, C)^2$ as desired.

3. (10 points)
The set of points (x, y) that satisfy the equation

$$x^2 + y^2 - 14y - 15 = 0$$

forms a circle. Find the center and radius of the circle.

Solution : We need only complete the square.

$$\begin{aligned}
&x^2 + y^2 - 14y - 15 = 0 \\
&x^2 + y^2 - 14y = 15 \\
&x^2 + y^2 - 14y + 7^2 = 15 + 7^2 \\
&x^2 + (y - 7)^2 = 64 \\
&(x - 0)^2 + (y - 7)^2 = 8^2.
\end{aligned}$$

This equation describes a circle of radius 8 centered at the point $(0, 7)$.

4. (20 points)
Let $Q = (1, 2)$ and $R = (-1, -1)$.

(a) Find the equation of the line through the points Q and R.

(b) Find the equation describing the set of points that lie on the perpendicular bisector of the segment with endpoints Q and R.

Solution :

(a) The slope of the line through Q and R

$$m = \frac{-1-2}{-1-1} = \frac{3}{2}.$$

So, the equation of the line will be $y - 2 = \frac{3}{2}(x - 1)$. Note we could have used the other point to write the equation as $y - (-1) = \frac{3}{2}(x - (-1))$. These both reduce to the same equation $y = \frac{3}{2}x + \frac{1}{2}$.

(b) There are two ways to solve this problem. The first is to use the fact that points on the perpendicular bisector are equidistant from Q and R. If we let (x, y) denote any point on the perpendicular bisector, then we get the equation

$$\begin{aligned}
&d(Q, (x, y)) = d(R, (x, y)) \\
&\sqrt{(x-1)^2 + (y-2)^2} = \sqrt{(x-(-1))^2 + (y-(-1))^2} \\
&\sqrt{(x-1)^2 + (y-2)^2} = \sqrt{(x+1)^2 + (y+1)^2} \\
&(x-1)^2 + (y-2)^2 = (x+1)^2 + (y+1)^2 \\
&(x^2 - 2x + 1) + (y^2 - 4y + 4) = (x^2 + 2x + 1) + (y^2 + 2y + 1) \\
&x^2 - 2x + y^2 - 4y + 5 = x^2 + 2x + y^2 + 2y + 2 \\
&-2x + y^2 - 4y + 5 = 2x + y^2 + 2y + 2 \\
&-2x - 4y + 5 = 2x + 2y + 2 \\
&-4x - 4y + 5 = 2y + 2 \\
&-4x - 4y + 3 = 2y \\
&-4x + 3 = 6y \\
&-\frac{2}{3}x + \frac{1}{2} = y.
\end{aligned}$$

The second approach involves using the fact that the perpendicular bisector must pass through the midpoint of the line segment with endpoints Q and R, and the slope of the perpendicular bisector must be the negative reciprocal of the slope of the line from part (a). The midpoint is

$$\left(\frac{1+(-1)}{2}, \frac{2+(-1)}{2}\right) = \left(0, \frac{1}{2}\right)$$

and the slope will be

$$-\frac{1}{\left(\frac{3}{2}\right)} = -\frac{2}{3}.$$

Combining this data into the point slope formula gives us the equation

$$y - \frac{1}{2} = -\frac{2}{3}(x - 0)$$

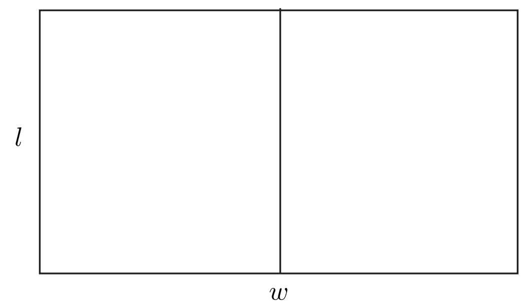
which reduces to the same equation obtained using the other approach.

5. (20 points)
   A person bitten by a zombie will turn into a zombie themselves. Leah is turned into a zombie at 11:00am and by 8:00am has bitten 13 people. Given that Leah can continue to bite people at this rate, find an equation describing how many people have turned into zombies as a direct result of Leah's bite after t hours.

   Solution : Our zombie friend Leah has bitten 0 people at 11:00, and 21 hours later has bitten 13 people. This corresponds to the two points $(0, 0)$ and $(21, 13)$. The equation of the line that passes through these points will be $z(t) = \frac{13}{21}t$.

6. (20 points)
A local farmer is interested in (quickly) constructing a fence consisting of two areas as seen below. One area will hold people bitten by zombies who have not yet turned and the other will hold full fledged zombies. He has 6 meters of fencing to construct the fence. Find the dimensions that will maximize the area enclosed.

Solution : We start with a picture.



The area is given by $A = lw$ and we know that the since the farmer only has 6 meters of fence we get a second equation $2l + 3w = 6$. Soling the second equation for l tells us that $l = 3 - \frac{3}{2}w$ and using this in the area equation we get the function of one variable

$$A(w) = \left(3 - \frac{3}{2}w\right) w = -\frac{3}{2}w^2 + 3w.$$

This is the equation of a parabola that opens down. Therefore, it must have a maximum at the vertex

$$x = -\frac{b}{2a} = -\frac{3}{2\left(-\frac{3}{2}\right)} = 1.$$

When $w = 1$ we see that $l = 3 - \frac{3}{2}(1) = \frac{3}{2}$.