(4 points) Name:____________________

Test 2
Fall 2006
MATH 111 Section 02
October 6, 2006

Directions : You have 50 minutes to complete all 5 problems on this exam. There are a possible 100 points to be earned. You may not use your book, notes, or any graphing/programmable calculator. Please be sure to show all pertinent work. *An incorrect answer with no work will receive no credit!* If any portion of the exam is unclear please come to me and I will elaborate provided I can do so without giving away the problem.

1. (10 points)
True or False. (2 points each)

   (a) There exists a polynomial $p(x)$ so that $p(x) \neq 0$ for any real number x.
   (b) There exists a polynomial $q(x)$ whose degree is odd so that $q(x) \neq 0$ for any real number x.
   (c) There exist polynomials $p(x)$ and $q(x)$ so that the rational polynomial $r(x) = p(x)/q(x)$ has no vertical asymptotes.
   (d) There exist polynomials $p(x)$ and $q(x)$ so that the rational polynomial $r(x) = p(x)/q(x)$ has no x-intercepts.
   (e) Given a polynomial $p(x)$, the number of real number solutions to the equation $p(x) = 0$ is equal to the degree of the polynomial.

Solution :

   (a) **TRUE** $f(x) = x^2 + 1$
   (b) **FALSE** Polynomials are unbounded (they have no horizontal asymptotes) and so if the degree is odd the polynomial must switch from negative to positive or vice versa. The sign change happens exactly when the graph crosses the x-axis.
   (c) **TRUE** $p(x) = x$ $q(x) = x^2 + 1$
   (d) **TRUE** $p(x) = x^2 + 1$ $q(x) = x$
   (e) **FALSE** $p(x) = x^2 + 1$ is degree 2 but has no real solutions.

2. (15 points)
Let $p(x) = -2x^6 - 2x^5 - 2x^4 - 3x^3 - 4x + 2$. What are all possibilities of the number of positive solutions, negative solutions, and complex solutions to the equation $p(x) = 0$?

Solution : We need only compute the variation of $p(x)$ and $p(-x)$. Recall the number of positive zeros of $p(x)$ is $\text{Var}(p(x))$ or $\text{Var}(p(x)) - 2$ or $\text{Var}(p(x)) - 4$ or $\text{Var}(p(x)) - 6$ etc... until this quantity becomes meaningless. The variation $\text{Var}(p(x))$ is the number of sign changes in the coefficients of the polynomial.

$$\text{Var}(p(x)) = 1.$$

So, the number of positive roots *must* be exactly 1 since we cannot subtract 2 and end up with anything meaningful (-1 roots?). To compute the possible number of negative roots we compute $\text{Var}(p(-x))$.

$$\text{Var}(p(-x)) = \text{Var}(-2x^6 + 2x^5 - 2x^4 + 3x^3 + 4x + 2) = 3$$

This tells us the number of negative roots is either 3 or 1. Our possibilities are then (keeping in mind that we must have a total of 6 roots)

Positive	1	1
Negative	3	1
Complex	2	4

3. (15 points)
Find a third degree polynomial $f(x)$ with the property that $f(0) = 1$, $f(-1) = 0$, $f(1) = 0$, and $f(2) = 0$.

Solution : A polynomial that is equal to zero when x is -1, 1, and 2 is

$$p(x) = (x - (-1))(x - 1)(x - 2).$$

However, $p(0) = (1)(-1)(-2) = 2$. We can fix this by multiplying $p(x)$ by $\frac{1}{2}$. This gives us the desired polynomial

$$f(x) = \frac{1}{2}(x + 1)(x - 1)(x - 2).$$

4. (20 points)
Do the following functions have horizontal or oblique asymptotes? If the asymptote is horizontal, what is it? [Do ***not*** compute the oblique asymptotes.]

(a)
$$f(x) = \frac{\pi x^2 - 3x + 1}{2x^2}$$

(b)
$$g(x) = \frac{3x^3 - x^2 + 5x + 6}{x^2 + 2x + 1}$$

(c)
$$h(x) = 3$$

(d)
$$k(x) = \frac{1380981x^2 + 1985x + 10^{100}}{0.00000000001x^3}$$

Solution :

(a) Horizontal asymptote $y = \frac{\pi}{2}$.

(b) Oblique asymptote (the degree of the numerator is larger than the degree of the denominator).

(c) Horizontal asymptote $y = 3$.

(d) Horizontal asymptote $y = 0$.

5. (36 points)

There are 6 parts to this problem; each part is worth 6 points. Consider the rational function:

$$f(x) = \frac{x^3 + x^2 - 14x - 24}{x^2 - x - 6}.$$

(a) Where are the x-intercepts if any?

(b) Where are the vertical tangent lines if any?

(c) Where are there holes in the graph if any?

(d) Where is the y-intercept if any?

(e) What is the horizontal/oblique asymptote?

(f) Sketch the graph.

Solution :

(a) The x-intercepts occur when the numerator is equal to zero. We know that if there are any rational values of x so that $x^3+x^2-14x-24 = 0$, then they must have the form

$$\pm\frac{\text{factor}(24)}{\text{factor}(1)} = \pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24.$$

After some trial and error we find that the solutions are $x = -3, -2, 4$ and so the numerator factors as

$$x^3 + x^2 - 14x - 24 = (x + 3)(x + 2)(x - 4).$$

The x intercepts are $x = -3, -2, 4$....or are they? Before we finalize this answer let's look at the next part.

(b) The vertical tangent lines occur when the denominator is zero.

$$x^2 - x - 6 = (x - 3)(x + 2).$$

So, at first glance, it appears that we have vertical asymptotes $x = -2$ and $x = 3$. But, wait a minute we said that $x = -2$ was also a x-intercept. It can't be both. Sure enough we see that we have factored the whole function as

$$f(x) = \frac{x^3 + x^2 - 14x - 24}{x^2 - x - 6}$$

$$= \frac{(x + 3)(x + 2)(x - 4)}{(x - 3)(x + 2)}$$

$$= \frac{(x + 3)(x - 4)}{x - 3} \text{ provided } x \neq -2.$$

Ah, we have a hole in the graph at $x = -2$. So, there is neither a x-intercept or a vertical asymptote at $x = -2$. So, the revised list of x-intercepts is $x = -3$ and $x = 4$ and we have one vertical asymptote when $x = 3$.

(c) As we found from the last part, there is only hole and it happens when $x = -2$.

(d) The y-intercept occurs when $x = 0$. So the y-intercept will be $y = f(0) = 4$.

(e) Since the degree of the numerator is larger than the degree of the denominator, there will be an oblique asymptote. To find it we use the division algorithm.

$$\begin{array}{r|rrrr}
 & & & x & +2 \\
\hline
x^2 - x - 6 & x^3 & +x^2 & -14x & -24 \\
 & -x^3 & +x^2 & +6x & \\
\hline
 & & 2x^2 & -8x & -24 \\
 & & -2x^2 & +2x & +12 \\
\hline
 & & & -6x & -12
\end{array}$$

This tells us that the oblique asymptote has equation $y = x + 2$.

(f)