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Test 1
Fall 2006
MATH 111 Section 04
September 7, 2006

Directions : You have 75 minutes to complete all 6 problems on this exam. There are a possible 100 points to be earned. You may not use your book, notes, or any graphing/programmable calculator. Please be sure to show all pertinent work. *An incorrect answer with no work will receive no credit!* If any portion of the exam is unclear please come to me and I will elaborate provided I can do so without giving away the problem.

1. (10 points)
   Solve the following inequalities, and express the solutions in terms of intervals if possible.

$$\frac{2}{2x+3} \le \frac{2}{x-5} \qquad\qquad 2|-11-7x|-2 > 10$$

Solution :

$$\frac{2}{2x+3} \le \frac{2}{x-5}$$

$$0 \le \frac{2}{x-5} - \frac{2}{2x+3}$$

$$0 \le \frac{2(2x+3)}{(x-5)(2x+3)} - \frac{2(x-5)}{(x-5)(2x+3)}$$

$$0 \le \frac{4x+6-2x+10}{(x-5)(2x+3)}$$

$$0 \le \frac{2x+16}{(x-5)(2x+3)}$$

The above fraction can only change signs at the points $x = -8$, $x = 5$, and $x = -3/2$ (when the top or bottom is zero). So, we need to only check the sign of the fraction at any points in the intervals $(-\infty, -8)$, $(-8, -3/2)$, $(-3/2, 5)$, and $(5, \infty)$. I get to choose any point in these intervals and so I will check the points $x = -10$, $x = -5$, $x = 0$, and $x = 10$.

If $x = -10$ then $\dfrac{2(-10)+16}{(-10-5)(2(-10)+3)} = \dfrac{-4}{(-15)(-17)} < 0$

If $x = -5$ then $\dfrac{2(-5)+16}{(-5-5)(2(-5)+3)} = \dfrac{6}{(-10)(-7)} > 0$

If $x = 0$ then $\dfrac{2(0)+16}{(0-5)(2(0)+3)} = \dfrac{16}{(-5)(3)} < 0$

If $x = 10$ then $\dfrac{2(10)+16}{(10-5)(2(10)+3)} = \dfrac{36}{(5)(23)} > 0$

We have shown that $0 < \frac{2x+16}{(x-5)(2x+3)}$ when x is in the set $(-8, -3/2) \cup (5, \infty)$ and it only remains to see if we include any of the endpoints (don't forget

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the problem asked about $\leq$ not just $<$). The only place the whole fraction is equal to zero is when the numerator is equal to zero. This is when $2x + 16 = 0$ so $x = -8$. Our solution is then $[-8, -3/2) \cup (5, \infty)$.

We now work on the second inequality.

$$2|-11 - 7x| - 2 > 10$$
$$2|-11 - 7x| > 12$$
$$|-11 - 7x| > 6$$
$$-11 - 7x < -6 \text{ OR } -11 - 7x > 6$$
$$-5 < 7x \text{ OR } -17 > 7x$$
$$-5/7 < x \text{ OR } -17/7 > x.$$

The solution set is $(-\infty, -17/7) \cup (-5/7, \infty)$.

2. (20 points)

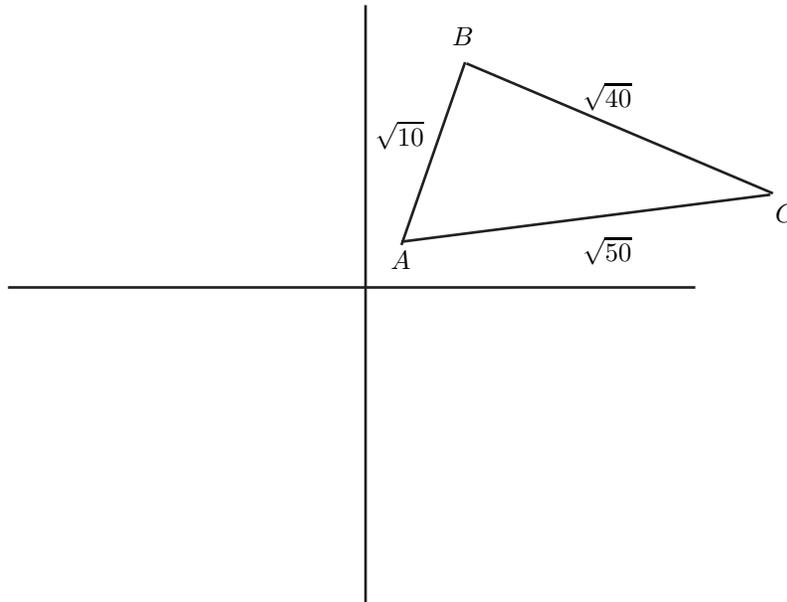Prove that the points $A = (1,1)$, $B = (2,4)$, and $C = (8,2)$ are the vertices of a right triangle.

Solution : We will show that the sides of the triangle obey the Pythagorean theorem. We begin by computing the distances between the points.

$$d(A,B) = \sqrt{(2-1)^2 + (4-1)^2} = \sqrt{10}$$
$$d(B,C) = \sqrt{(8-2)^2 + (2-4)^2} = \sqrt{40}$$
$$d(A,C) = \sqrt{(8-1)^2 + (2-1)^2} = \sqrt{50}.$$

Sure enough $d(A,B)^2 + d(B,C)^2 = d(A,C)^2$. So, it's a right triangle. Thanks Pythagoras!

3. (10 points)

   The set of points (x, y) that satisfy the equation

   $$x^2 + y^2 - 10x + 18 = 0$$

   forms a circle. Find the center and radius of the circle.

   Solution : To do this, we need only compete the square.

   $$
   \begin{aligned}
   x^2 + y^2 - 10x + 18 &= 0 \\
   (x^2 - 10x) + y^2 &= -18 \\
   (x^2 - 10x + 5^2) + y^2 &= -18 + 5^2 \\
   (x - 5)^2 + (y - 0)^2 &= 7.
   \end{aligned}
   $$

   This is the equation of a circle whose center is at the point $(5, 0)$ and the radius is $\sqrt{7}$.

4. (20 points)

Let $Q = (1, 2)$ and $R = (-1, -1)$.

(a) Find the equation of the line ℓ through the points Q and R.

(b) Find the equation of the line ℓ_2 through the midpoint of the line segment with endpoints Q and R.

(c) Find the x and y intercepts of ℓ_2.

Solution :

(a) The slope of the line will be $m = (-1 - 2)/(-1 - 1) = 3/2$. Using the point slope formula $y - y_1 = m(x - x_1)$ we get the equation $y - 2 = 3/2(x - 1)$ which reduces to $y = 3/2x + 1/2$. Notice we could have used the other point in the point slope equation to get the same result: $y - (-1) = 3/2(x - (-1))$ which reduces to $y = 3/2x + 1/2$. In either case $\ell(x) = 3/2x + 1/2$.

(b) There are two ways to do this problem. We can either use the fact that the distance between any point on the line ℓ_2 and Q is the same as the distance between the same point on ℓ_2 and R, or we can compute the midpoint and find the new slope.

   i. Let's start with the distance equals the distance approach.

$$d(Q, (x, y)) = d(R, (x, y))$$
$$\sqrt{(x - 1)^2 + (y - 2)^2} = \sqrt{(x - (-1))^2 + (y - (-1))^2}$$
$$\sqrt{x^2 - 2x + 1 + y^2 - 4y + 4} = \sqrt{x^2 + 2x + 1 + y^2 + 2y + 1}$$
$$x^2 - 2x + 1 + y^2 - 4y + 4 = x^2 + 2x + 1 + y^2 + 2y + 1$$
$$-2x + 1 + y^2 - 4y + 4 = 2x + 1 + y^2 + 2y + 1$$
$$-2x + 1 - 4y + 4 = 2x + 1 + 2y + 1$$
$$-2x - 4y + 5 = 2x + 2y + 2$$
$$-2x + 5 - 2x - 2 = 2y + 4y$$
$$-4x + 3 = 6y$$
$$-2/3x + 1/2 = y.$$

   ii. Now, let's find the midpoint and use the fact that the slope of ℓ_2 is the negative reciprocal of the slope of ℓ since they are perpendicular. The midpoint is $M = ((1 + (-1))/2, (2 + (-1))/2) = (0, 1/2)$. The slope of ℓ_2 is $-2/3$ and so the equation of the line ℓ_2 is $y - 1/2 = -2/3(x - 0)$ which reduces to $y = -2/3x + 1/2$, exactly what we got before. In either case $\ell_2(x) = -2/3x + 1/2$.

(c) The x intercepts of ℓ_2 occur when $y = 0$. So, this happens when $0 = -2/3x + 1/2$ or when $x = 3/4$.

5. (20 points)

A person bitten by a zombie will turn into a zombie themselves. Bud is turned into a zombie at 1:00am and by 6:00am has bitten 12 people. Given that Bud can continue to bite people at this rate, find an equation describing how many people have turned into zombies as a direct result of a Bud bite after t hours.

Solution : That "Bud bite" pun still makes me chuckle. Anyhoo, at the beginning Bud has bitten 0 people so this corresponds to a point $(0, 0)$ where the first zero indicates how much time has passed and the second how many people Bud has bitten by that time. Five hours later Bud has bitten 12 folks. So, this corresponds to the point $5(5, 12)$. We now have two points, which is plenty to describe a line. The line will have slope $m = (12 - 0)/(5 - 0) = 12/5$ and so it's equation will be $z(t) = \frac{12}{5}t$.

6. (20 points)

A farmer (Jim) is interested in (quickly) constructing a fence consisting of two areas. One area will hold people bitten by zombies who have not yet turned and the other will hold full fledged zombies. He has 1000 yards of fencing to construct the fence. Find the dimensions that will maximize the area enclosed.

Solution : We start with a picture.
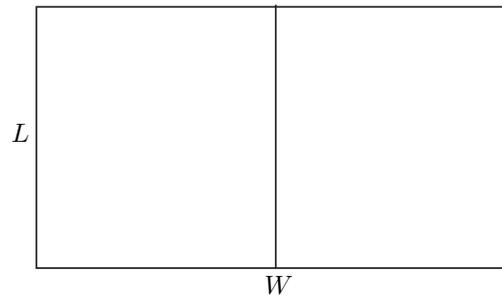
The area of this enclosure will be $A = LW$ and this is the quantity we want to maximize. Alas! This is a function of two variables and we don't know how to deal with them. But, we know that Jim only has 1000 yards of fence. This gives us the equation $2W + 3L = 1000$ and we can solve this for L to get $W = 500 - 3/2L$. Now, we can rewrite the area equation as $A = L(500 - 3/2L) = -3/2L^2 + 500L$. This is a parabola that opens down and so it has a maximum at its vertex. The x coordinate of the vertex will be $-500/-3 = 500/3$ and so we deduce that the area is maximized when $L = 500/3$ and $W = 500 - 3/2(500/3) = 250$.