Definition 1 *Given a function $f(x)$, we say the limit of $f(x)$ as x approaches a is equal to L, and we write*

$$\lim_{x \to a} f(x) = L,$$

provided given $\varepsilon > 0$, there exists $\delta > 0$ so that $|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$.

The idea behind the definition of a limit is that "locally near the point a, the function $f(x)$ looks like L." Let me see if I can make any more sense of that. Let's work with a specific example. Suppose that the function we have is $f(x) = x^3 + 2$ and we are interested in computing the limit as x approaches zero. Well, if there is any justice in the world, the value of this limit should be 2, but why? Let's pause and think about what the function $f(x)$ looks like really near the origin. Suppose that I consider the value of $f(x)$ when I have x restricted to the set (-10,10). In this case the function values (i.e., the y-values of the graph) range from $-10^3 + 2 = -998$ to $10^2 + 2 = 1002$. That's quite a range. With a range that diverse I'm really reluctant to say anything about what the y-value of the graph is near zero. So, okay, lets get a little closer to zero. Let's now only look at the x values in the set $(-5, 5)$ and see what the y values look like. Better, but still no good. The y values now fall somewhere in the range (-123,127). I'm still not comfortable making predictions about $f(x)$ near zero when the range is that large. Let me try once more. Finally, suppose that the x-values live in the set $(-0.1, 0.1)$. Then, the y-values must live in the range $(1.999, 2.001)$. That's *much* better. Do you see how the range in the y values is affected by the change in the range of x-values?

If I claim that $\lim_{x \to 0} x^3 + 2 = 2$, then I need to be prepared if someone asks me to prove it. There may be some lunatic walking down the street at 4am that stops me and demands that I show to him that I can guarantee that $x^3 + 2$ can always be made within 5 of the limit value 2. What I need to do is then find for him a range of x-values so that the corresponding range of y-values is always in the interval $(2 - 5, 2 + 5) = (-3, 7)$. So, as our previous work shows, we can casually point out to the madman that if we select all of our x-values in the set $(-0.1, 0.1)$, then all of the y-values must lie in the interval $(1.999, 2.001)$. In this case the numbers in the y-value range differ from 2 by no more than 0.001, which is certainly a lot better than 5. Crazy guy leaves satisfied.

Enter crazy person #2. They demand that I show them that I can guarantee that the value of $x^3 + 2$ lives no more than 0.000001 units away from 2 when the x-values are really close to $x = 0$. Restating the question: I want to find x-values so that I can be guaranteed that when used in the function $x^3 + 2$, the result lives somewhere in the set $(2 - 0.000001, 2 + 0.000001) = (1.999999, 2.000001)$. Check it out, if x lives in the set $(-0.01, 0.01)$, then $x^3 + 2$ must live in $(1.999999, 2.000001)$.

Finally, one last crazy person. They walk up to me and ask me the same type of question *but*, they don't tell me what the y-value range is. They say, that they are holding some tiny number c in their hand and they want me to show them that $x^3 + 2$ can be made to live in $(2 - c, 2 + c)$ without telling me c! Man, I had better cover all my bases. I can't just tell him to use an interval like $(-0.1, 0.1)$ because c may be smaller than 0.0001. I can't even safely guess that $(-0.01, 0.01)$ will work since c may be smaller than 0.000001. I need to be more fancy. My interval had better involve c. Check it out. Suppose I answered him in kind and said, "Okay, how about $(-c^{1/3}, c^{1/3})$". What happens to the y-values in this interval? Well, the function $x^3 + 2$ is an increasing function. So, the smallest it can possibly be in this interval is $(-c^{1/3})^3 + 2 = -c + 2$. Similarly, the largest it can be in this interval is $(c^{1/3})^3 + 2 = c + 2$. So, if the x-values are restricted to the set $(-c^{1/3}, c^{1/3})$, then the y-values must live in the set $(2 - c, 2 + c)$.

In the language of the definition above, $\varepsilon = c$ and $\delta = c^{1/3}$. The thing to notice is that in order to find δ we had to think about how to undo the interval involving c via the function $x^3 + 2$. What we did was use a type of inverse function. Notice that $x^3 + 2$ "looks like" the function $f(x) = x^3$. How do we undo what has been done with f? We use the inverse function. The inverse of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$. This is really how we found the interval $(-c^{1/3}, c^{1/3})$.