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Test 1
Fall 2006
MATH 121 Section 02
September 12, 2006

Directions : You have 50 minutes to complete all 5 problems on this exam.
There are a possible 100 points to be earned. You may not use your book, notes,
or any graphing/programmable calculator. Please be sure to show all pertinent
work. *An incorrect answer with no work will receive no credit!* If any portion
of the exam is unclear please come to me and I will elaborate provided I can do
so without giving away the problem.

1. (40 points)

Compute the following limits.

a) $\displaystyle\lim_{x\to 3} \frac{x^2 - 2x - 3}{x^2 - x - 6}$ 　　　　 b) $\displaystyle\lim_{h\to 0} \frac{\sqrt{1+h} - 1}{h}$

c) $\displaystyle\lim_{x\to 1} \frac{\sqrt{x} - 1}{x - 1}$ 　　　　 d) $\displaystyle\lim_{x\to a} \pi$

Solution :

(a)

$$\lim_{x\to 3} \frac{x^2 - 2x - 3}{x^2 - x - 6} = \lim_{x\to 3} \frac{(x+1)(x-3)}{(x-3)(x+2)}$$
$$= \lim_{x\to 3} \frac{x+1}{x+2}$$
$$= \frac{4}{5}.$$

(b)

$$\lim_{h\to 0} \frac{\sqrt{1+h} - 1}{h} = \lim_{h\to 0} \frac{\sqrt{1+h} - 1}{h} \left(\frac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1} \right)$$
$$= \lim_{h\to 0} \frac{h}{h(\sqrt{1+h} + 1)}$$
$$= \lim_{h\to 0} \frac{1}{\sqrt{1+h} + 1}$$
$$= \frac{1}{2}.$$

(c)

$$\lim_{x\to 1} \frac{\sqrt{1} - 1}{x - 1} = \lim_{x\to 1} \frac{\sqrt{x} - 1}{x - 1} \frac{\sqrt{x} + 1}{\sqrt{x} + 1}$$
$$= \lim_{x\to 1} \frac{x - 1}{(x - 1)(\sqrt{x} + 1)}$$
$$= \lim_{x\to 1} \frac{1}{\sqrt{x} + 1}$$
$$= \frac{1}{2}.$$

(d)

$$\lim_{x\to a} \pi = \pi.$$

2. (10 points)
   Prove that there is a number that is exactly one more than its cube.
   (Hint: The Intermediate Value Theorem)

   Solution : We are to show that there is a number x so that $x = x^3 + 1$.
   Rewriting this we see that we need to show that there is an x so that
   $x^3 - x + 1 = 0$. The hint was to use the intermediate value theorem.
   Recall that the intermediate value theorem says that if $f(x)$ is a continuous
   function on the interval $[a, b]$ then for each constant c in between $f(a)$
   and $f(b)$ there exists some value x_c in $[a, b]$ so that $f(x_c) = c$. A more
   direct application is that if f is a continuous function and if $f(a) < 0$
   and $f(b) > 0$, then there must be a value x_0 in between a and b so that
   $f(x_0) = 0$. Define $f(x) = x^3 - x + 1$ and observe that f is continuous
   since it is a polynomial. Our goal is to prove that there is a value x_0 so
   that $f(x_0) = 0$. Noticing that $f(-2) = -5 < 0$ and $f(0) = 1 > 0$, the
   intermediate value theorem guarantees that there is a value x_0 in between
   -2 and 0 so that $f(x_0) = 0$. This is what we wanted to show since
   $f(x_0) = 0$ can bve rewritten as $x_0 = (x_0)^3 + 1$.

3. (20 points)

A zombie lurches toward you in the night and hands you the function $z(x) = \sqrt{x}$. He demands that you find for him a number δ so that if x is any number whose distance from 4 is less than δ, then the distance from 2 to $z(x)$ is less than 0.1.

Solution : This zombie really knows his calculus! Maybe it's the re-animated corpse of Newton. Anyhoo, recall the definition of the limit. The limit of the function $f(x)$ as x approaches a is equal to L, written $\lim_{x \to a} f(x) = L$, provided given an $\varepsilon > 0$ there exists a $\delta > 0$ so that $|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$. The zombie Newton has handed us a value of $\varepsilon = 0.1$ and wants to find a value of δ so that $|\sqrt{x} - 2| < 0.1$ whenever $|x - 4| < \delta$.
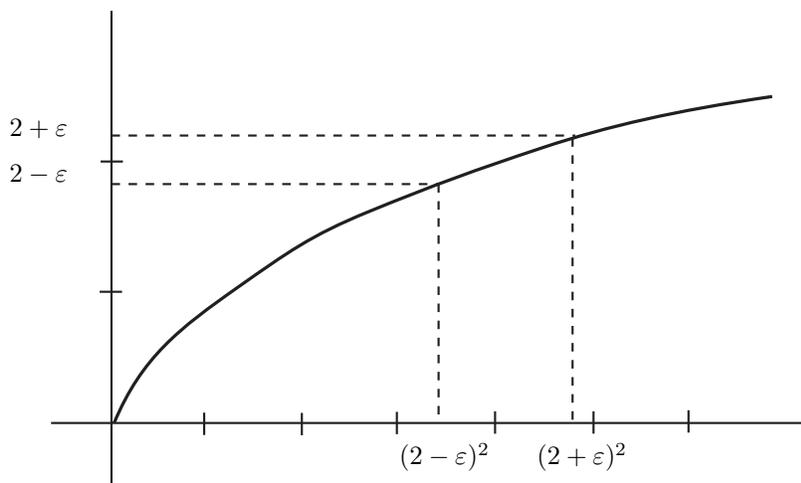
The choices of δ will be $4 - (2 - 0.1)^2$ and $(2 + 0.1)^2) - 4$ and we will choose whichever is smallest.

$$4 - (2 - 0.1)^2 = 4 - (1.9)^2 = 4 - 3.61 = 0.39$$
$$(2 + 0.1)^2 - 4 = (2.1)^2 - 4 = 4.41 - 4 = 0.41.$$

So, we let $\delta = 0.39$. This will be easier to understand with a picture. I will put up a picture as soon as the software gets here.

4. (10 points)
   Find a value for a so the following function is continuous for all values of x in $(-\infty, \infty)$. If no such value exists, tell me why.

$$f(x) := \begin{cases} x^2 + 1 & \text{if } x < 1 \\ (x^2 + a)/x & \text{if } x \geq 1 \end{cases}$$

Solution : Remember that a function $f(x)$ is continuous at the point $x = b$ provided $f(b)$ is defined, $\lim_{x \to b} f(x)$ exists, and $\lim_{x \to b} f(x) = f(b)$. Each of the components of the function f are continuous where they are defined. If $x < 1$ then $f(x) = x^2 + 1$ which is a polynomial and so it is continuous everywhere. If $x > 1$, then $f(x) = (x^2 + a)/x$ is continuous since it's only potential discontinuity occurs when $x = 0$ (and $x > 1$). The only place where the function may fail to be continuous is when $x = 1$. We know that $f(1) = (1 + a)$ and so it is defined when $x = 1$. It remains to show that the limit exists and is equal to $f(1)$.

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{x^2 + a}{x} = 1 + a \text{ and}$$
$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 + 1 = 2.$$

This limit only exists when $1 + a = 2$ an this means that $a = 1$. So, the continuous function is

$$f(x) := \begin{cases} x^2 + 1 & \text{if } x < 1 \\ (x^2 + 1)/x & \text{if } x \geq 1 \end{cases}$$

5. (20 points)

   Find the equation of the line tangent to the curve $f(x) = x^2 + 2x + 4$ at the point $(1, 7)$.

   Solution : The equation of the tangent line to the curve $y = f(x)$ at the point $x = 1$ will be $\ell(x) = f'(a)(x - a) + f(a)$. We need to compute the derivative at the point $x = 1$. We have $f'(x) = 2x + 2$ and so $f'(1) = 4$. The equation of the line is then

   $$\ell(x) = 4(x - 1) + 7.$$