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Test 3
Fall 2006
MATH 121 Section 02
October 31 + 2, 2006 = November 2, 2006

Directions : You have 50 minutes to complete all 6 problems on this exam. There are a possible 100 points to be earned. You may not use your book, notes, or any graphing/programmable calculator. Please be sure to show all pertinent work. *An incorrect answer with no work will receive no credit!* If any portion of the exam is unclear please come to me and I will elaborate provided I can do so without giving away the problem.

1. (15 points)
Let $f(x) = x^3 + 6x^2 + 9x + 3$. Find the absolute maximum and minimum values of $f(x)$ on the interval $[-4, 1]$.

Solution : First we find the critical points.

$$f'(x) = 3x^2 + 12x + 9 = 3(x^2 + 4x + 3) = 3(x + 1)(x + 3).$$

The only critical points occur when $f'(x) = 0$ since $f'(x)$ is defined everywhere. We see that $f'(x) = 0$ when $x = -1$ and $x = -3$ and both of these points live inside the interval $[-4, 1]$. We now check all of the critical points in $[-4, 1]$ as well as the endpoints to find which is the largest and which is the smallest.

$$\begin{aligned}
\textbf{Min} f(-4) &= -64 + 6(16) - 36 + 3 = -64 + 96 - 36 + 3 = -1 \\
f(-3) &= -27 + 54 - 27 + 3 = 3 \\
\textbf{Min } f(-1) &= -1 + 6 - 9 + 3 = -1 \\
\textbf{Max } f(1) &= 1 + 6 + 9 + 3 = 19.
\end{aligned}$$

2. (15 points)
   Let $g(x) = (x^2 - 1)^3$. Find the absolute maximum and minimum values of $g(x)$ on the interval $[-1, 2]$

   Solution : First we find the critical points.

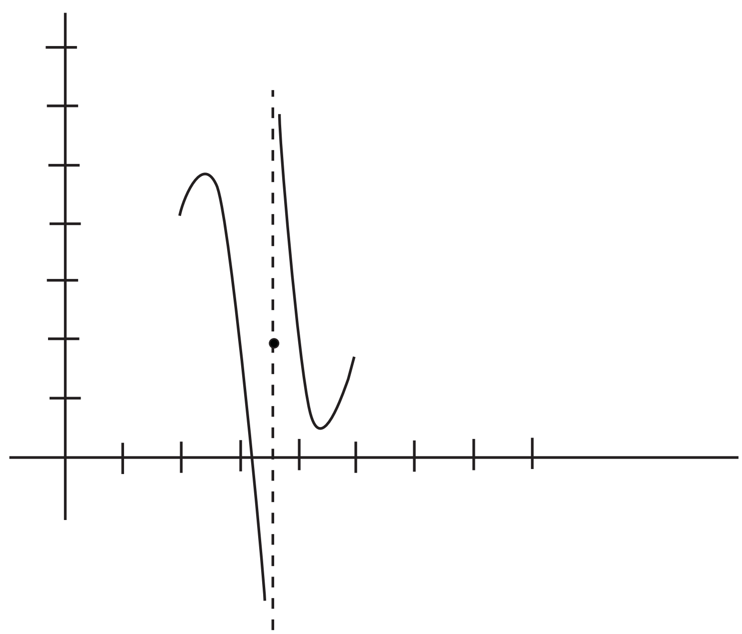
$$g'(x) = 3(x^2 - 1)^2(2x)$$

   The only critical points occur when $g'(x) = 0$ since $g'(x)$ is defined everywhere. We see that $g'(x) = 0$ when $x = 0$ and $x = \pm 1$ and all of these points live inside the interval $[-1, 2]$. We now check all of the critical points in $[-1, 2]$ as well as the endpoints to find which is the largest and which is the smallest.

$$g(-1) = 0$$
$$\textbf{Min } g(0) = -1$$
$$g(1) = 0$$
$$\textbf{Max } g(2) = 27.$$

3. (15 points)
Sketch the graph of a function on the interval $[2, 5]$ that is defined at all points in $[2, 5]$, has a local maximum and a local minimum, and has neither an absolute maximum or absolute minimum. [Hint: Can it be continuous?]

Solution : The graph cannot be continuous on $[2, 5]$ because if it was it would achieve both an absolute maximum and an absolute minimum. I have drawn a graph with a vertical asymptote at some point in between 2 and 5 so that the graph is unbounded near the asymptote. Since it needs to be defined at all points in $[2, 5]$ I have defined the value of the function to be 2 at the asymptote.



4. (10 points)
State Rolle's Theorem and the Mean Value Theorem.

Solution :
Rolle's Theorem : Let f be a function that satisfies the following three hypothesis :

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

Mean Value Theorem : Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$f(b) - f(a) = f'(x)(b - a)$$

5. (20 points)
Use the Intermediate Value Theorem and Rolle's Theorem to show that there is exactly one solution to the equation $2x^3 + 3x + \sin(x) = 0$.

Solution : Define $f(x) = 2x^3 + 3x + \sin(x)$. Notice that $f'(x)$ is continuous since it is the sum of continuous functions and that $f(\pi) = 2\pi^3 + 3\pi > 0$ and $f(-\pi) = -2\pi^3 - 3\pi < 0$. The intermediate value theorem asserts that $f(c) = 0$ for some c in between $-\pi$ and π (in fact $f(0) = 0$). We are now guaranteed that there is at least one solution. It remains to show that there is only one. Suppose, to the contrary, that there are two values c and b so that $f(c) = 0$ and $f(b) = 0$. Then, the mean value theorem states that there must be a point in between c and b so that the derivative vanishes. However, $f'(x) = 6x^2 + 3 + \cos(x) > 0$ for all x and so the derivative cannot vanish. This is a contradiction. So, there must only be a single point c so that $f(c) = 0$.

6. (25 points)
Below is the graph of the first derivative of a function f.

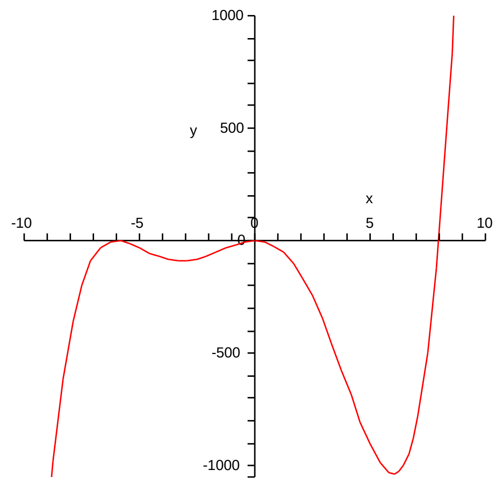
(a) Find the intervals where f is increasing and the intervals where f is decreasing.

(b) Locate and identify all critical points as local maxima, local minima, or neither.

(c) Find the intervals where f is concave up and the intervals where f is concave down.

(d) Identify any inflection points.

(e) Sketch the graph of a function f whose derivative is shown below.



Solution :

(a) Increasing $(8, \infty)$
Decreasing $(-\infty, -5) \cup (-5, 0) \cup (0, 8)$

(b) Critical points
$x = -5$ neither,
$x = 0$ neither,
$x = 8$ min.

(c) Concave up $(-\infty, -5) \cup (-3, 0) \cup (6, \infty)$
Concave down $(-5, -3) \cup (0, 6)$.

(d) Inflection points $x = -5, -3, 0, 6$.

(e)